VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2007

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

STUDENT NUMBER

									Letter
Figures									
Words									

MATHEMATICAL METHODS (CAS)

Written examination 2

Monday 12 November 2007

Reading time: 11.45 am to 12.00 noon (15 minutes)
Writing time: 12.00 noon to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
1	22	22	22
2	5	5	58
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching, one bound reference, one approved CAS calculator (memory DOES NOT need to be cleared) and, if desired, one scientific calculator. For approved computer based CAS, their full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or white out liquid/tape.

Materials supplied
- Question and answer book of 23 pages with a detachable sheet of miscellaneous formulas in the centrefold.
- Answer sheet for multiple-choice questions.

Instructions
- Detach the formula sheet from the centre of this book during reading time.
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION 1

Instructions for Section 1

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1, an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.

Question 1

The linear function $f: D \to R,\ f(x) = 6 - 2x$ has range [–4, 12].
The domain D is

A. [–3, 5]
B. [–5, 3]
C. R
D. [–14, 18]
E. [–18, 14]

Question 2

Let $g(x) = x^2 + 2x - 3$ and $f(x) = e^{2x+3}$.
Then $f(g(x))$ is given by

A. $e^{4x+6} + 2\,e^{2x+3} - 3$
B. $2x^2 + 4x - 6$
C. e^{2x^2+4x+9}
D. e^{2x^2+4x-3}
E. e^{2x^2+4x-6}

Question 3

If $y = \log_a(7x - b) + 3$, then x is equal to

A. $\frac{1}{7}a^{y-3} + b$
B. $\frac{1}{7}(a^{y} - 3) + b$
C. $\frac{1}{7}(a^{y-3} + b)$
D. $a^{y-3} - \frac{b}{7}$
E. $\frac{y-3}{\log_a(7-b)}$

Question 4

The average rate of change of the function with rule $f(x) = x^3 - \sqrt{x + 1}$ between $x = 0$ and $x = 3$ is

A. 0

B. 12

C. $\frac{26}{3}$

D. $\frac{25}{3}$

E. 8

Question 5

The simultaneous linear equations

$$mx + 12y = 24$$
$$3x + my = m$$

have a unique solution only for

A. $m = 6$ or $m = -6$

B. $m = 12$ or $m = 3$

C. $m \in R \setminus \{-6, 6\}$

D. $m = 2$ or $m = 1$

E. $m \in R \setminus \{-12, -3\}$

Question 6

The range of the function $f: \left[0, \frac{\pi}{3}\right) \to R,\ f(x) = 3\left|\sin(2x) - 1\right| + 2$ is

A. $[2, 5)$

B. $[2, 5]$

C. $\left[5 - \frac{3\sqrt{3}}{2}, 5\right)$

D. $\left(5 - \frac{3\sqrt{3}}{2}, 5\right]$

E. $(2.5, 5]$

Question 7

The random variable X has a normal distribution with mean 11 and standard deviation 0.25.

If the random variable Z has the standard normal distribution, then the probability that X is less than 10.5 is equal to

A. $\Pr(Z > 2)$

B. $\Pr(Z < -1.5)$

C. $\Pr(Z < 1)$

D. $\Pr(Z \geq 1.5)$

E. $\Pr(Z < -4)$

Question 8

Which one of the following is **not** true about the function $f: R \to R,\ f(x) = |2x + 4|$?

A. The graph of f is continuous everywhere.

B. The graph of f' is continuous everywhere.

C. $f(x) \geq 0$ for all values of x

D. $f'(x) = 2$ for all $x > 0$

E. $f'(x) = -2$ for all $x < -2$

Question 9

Let $k = \int_{-2}^{-1} \frac{1}{x}\,dx$, then e^k is equal to

A. $\log_e(2)$

B. 1

C. 2

D. e

E. $\frac{1}{2}$

Question 10

The graph of $y = kx - 3$ intersects the graph of $y = x^2 + 8x$ at two distinct points for

A. $k = 11$

B. $k > 8 + 2\sqrt{3}$ or $k < 8 - 2\sqrt{3}$

C. $5 \leq k \leq 6$

D. $8 - 2\sqrt{3} \leq k \leq 8 + 2\sqrt{3}$

E. $k = 5$

Question 11

The solution set of the equation $e^{4x} - 5e^{2x} + 4 = 0$ over R is

A. $\{1, 4\}$

B. $\{-4, -1\}$

C. $\{-2, -1, 1, 2\}$

D. $\{-\log_e(2), 0, \log_e(2)\}$

E. $\{0, \log_e(2)\}$

Question 12

Let $f: R \rightarrow R$ be a differentiable function such that

- $f'(3) = 0$
- $f'(x) < 0$ when $x < 3$ and when $x > 3$

When $x = 3$, the graph of f has a

A. local minimum.

B. local maximum.

C. stationary point of inflection.

D. point of discontinuity.

E. gradient of 3.

Question 13

For the graph of $y = 4x^3 + 27x^2 - 30x + 10$ the subset of R for which the gradient is negative is given by the interval

A. $(0.5, 5.0)$

B. $(-4.99, 0.51)$

C. $\left(-\infty, \frac{1}{2}\right)$

D. $\left(-5, \frac{1}{2}\right)$

E. $(2.25, \infty)$

Question 14

The maximal domain D of the function $f: D \rightarrow R$ with rule

$$f(x) = \log_e(|x - 3|) + 6 \text{ is}$$

A. $R \setminus \{3\}$

B. $(3, \infty)$

C. R

D. $(-3, \infty)$

E. $(-\infty, 3)$

Question 15

The graph of the function f: $[0, \infty) \to R$ where $f(x) = 3x^{\frac{5}{2}}$ is reflected in the x-axis and then translated 3 units to the right and 4 units down.

The equation of the new graph is

A. $y = 3(x - 3)^{\frac{5}{2}} + 4$

B. $y = -3(x - 3)^{\frac{5}{2}} - 4$

C. $y = -3(x + 3)^{\frac{5}{2}} - 1$

D. $y = -3(x - 4)^{\frac{5}{2}} + 3$

E. $y = 3(x - 4)^{\frac{5}{2}} + 3$

Question 16

If a random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if} \quad x \in [0,\ 2] \\ 0 & \text{otherwise} \end{cases}$$

then $E(X)$ is equal to

A. $\frac{1}{2}$

B. 1

C. $\frac{4}{3}$

D. $\frac{2}{3}$

E. 2

Question 17

The function f satisfies the functional equation $f(f(x)) = x$ for the maximal domain of f.

The rule for the function is

A. $f(x) = x + 1$

B. $f(x) = x - 1$

C. $f(x) = \dfrac{x-1}{x+1}$

D. $f(x) = \log_e(x)$

E. $f(x) = \dfrac{x+1}{x-1}$

Question 18

The heights of the children in a queue for an amusement park ride are normally distributed with mean 130 cm and standard deviation 2.7 cm. 35% of the children are not allowed to go on the ride because they are too short.

The minimum acceptable height correct to the nearest centimetre is

A. 126

B. 127

C. 128

D. 129

E. 130

Question 19

The discrete random variable X has probability distribution as given in the table. The mean of X is 5.

x	0	2	4	6	8
$\Pr(X = x)$	a	0.2	0.2	0.3	b

The values of a and b are

A. $a = 0.05$ and $b = 0.25$

B. $a = 0.1$ and $b = 0.29$

C. $a = 0.2$ and $b = 0.9$

D. $a = 0.3$ and $b = 0$

E. $a = 0$ and $b = 0.3$

Question 20

The average value of the function $y = \tan(2x)$ over the interval $\left[0, \frac{\pi}{8}\right]$ is

A. $\frac{2}{\pi}\log_e(2)$

B. $\frac{\pi}{4}$

C. $\frac{1}{2}$

D. $\frac{4}{\pi}\log_e 2$

E. $\frac{8}{\pi}$

Question 21

$\{x: \cos^2(x) + 2\cos(x) = 0\} =$

A. $\{x: \cos(x) = 0\}$

B. $\left\{x: \cos(x) = -\frac{1}{2}\right\}$

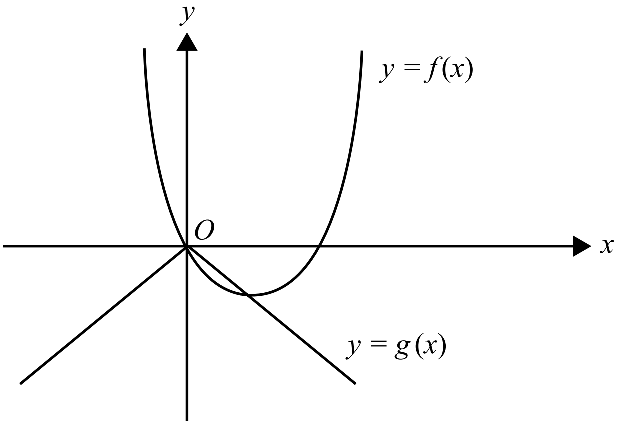
C. $\left\{x: \cos(x) = \frac{1}{2}\right\}$

D. $\{x: \cos(x) = 0\} \cup \left\{x: \cos(x) = -\frac{1}{2}\right\}$

E. $\left\{x: \cos(x) = \frac{1}{2}\right\} \cup \left\{x: \cos(x) = -\frac{1}{2}\right\}$

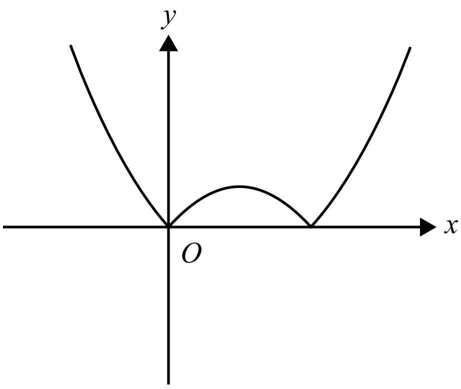
Question 22

The graphs of $y = f(x)$ and $y = g(x)$ are as shown.

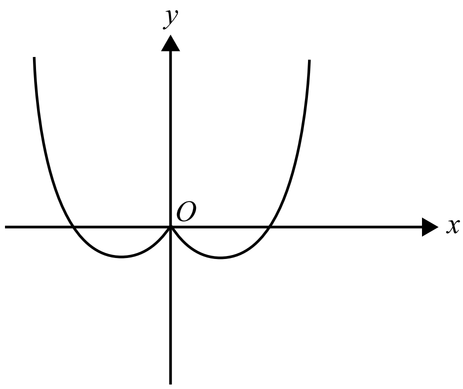


The graph of $y = f(g(x))$ is best represented by

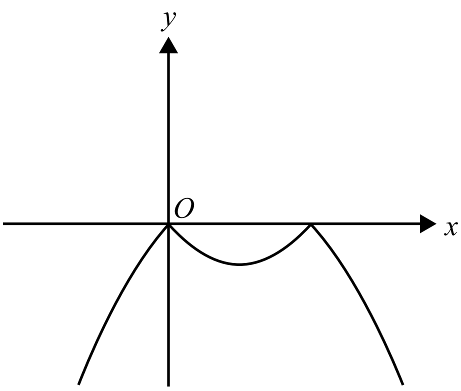
A.



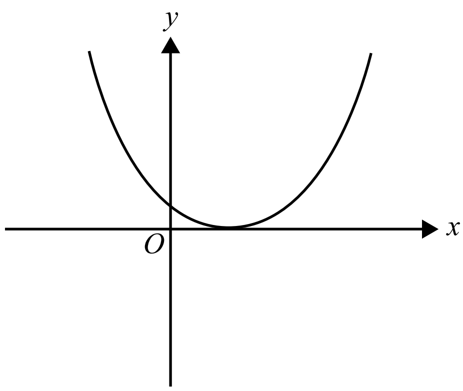
B.



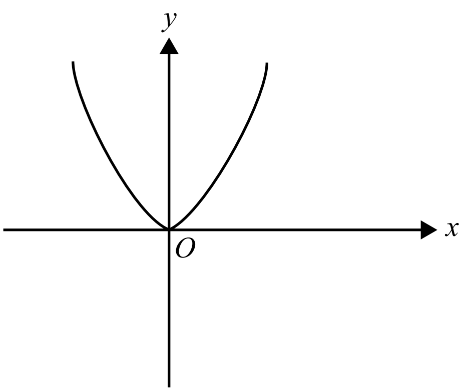
C.



D.



E.



END OF SECTION 1

Working space

SECTION 2

Instructions for Section 2

Answer **all** questions in the spaces provided.
A decimal approximation will not be accepted if an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1

A dog food company manufactures dog food in cylindrical cans of volume V cm^3.

The height of the can is h cm and the radius of the can is r cm, where $\frac{V^{\frac{1}{3}}}{10} \leq r \leq \frac{5V^{\frac{1}{3}}}{9}$.

a. Express h in terms of r and V.

1 mark

b. Show that the total surface area, A cm^2, of the can is given by $A = \frac{2V}{r} + 2\pi r^2$.

2 marks

c. Find, in terms of V, the value of r so that the surface area of the can is a minimum. (You are not required to justify the nature of the stationary point.)

2 marks

d. Find, in terms of V, the maximum possible surface area of the can.

2 marks

Total 7 marks

Question 2

Tasmania Jones is attempting to recover the lost Zambeji diamond. The diamond is buried at a point 4 km into Death Gorge, which is infested with savage insects. In order to recover the diamond, Tasmania will need to run into the gorge, dig up the diamond and return the same way that he came.

The concentration of insects in the gorge is a **continuous** function of time. The concentration C, insects per square metre, is given by

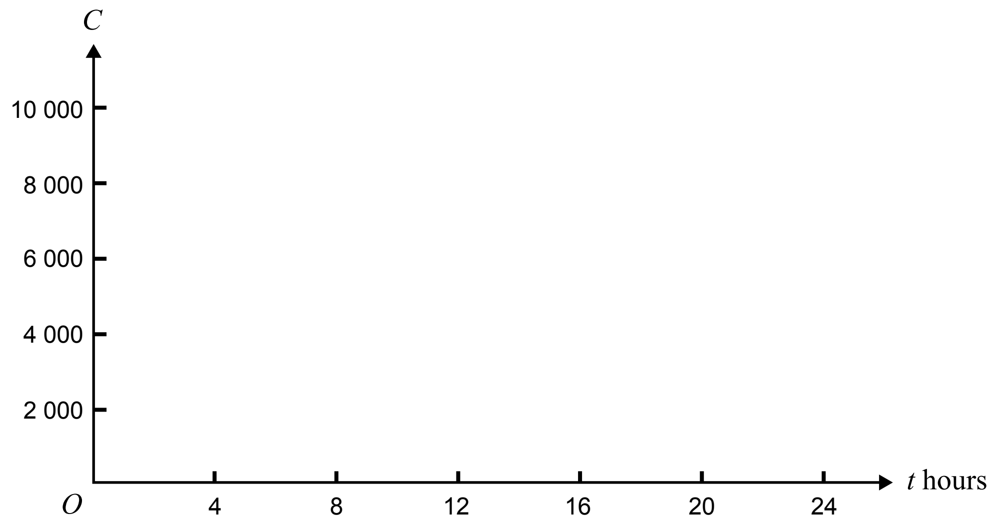
$$C(t) = \begin{cases} 1000\left(\cos\left(\dfrac{\pi(t-8)}{2}\right)+2\right)^2 - 1000 & 8 \le t \le 16 \\ m & 0 \le t < 8 \text{ or } 16 < t \le 24 \end{cases}$$

where t is the number of hours after midnight and m is a real constant.

a. What is the value of m?

1 mark

b. Sketch the graph of C for $0 \le t \le 24$



3 marks

c. What is the minimum concentration of insects and at what value(s) of t does that occur?

2 marks

The insects infesting the gorge are known to be deadly if their concentration is more than 1250 insects per square metre.

d. At what time after midnight does the concentration of insects first stop being deadly?

1 mark

e. During a 24-hour period, what is the total length of time for which the concentration of insects is less than 1250 insects per square metre?

2 marks

Due to the uneven surface of the gorge, the time, T minutes, that Tasmania will take to run x km into the gorge is given by $T = p(q^x - 1)$, where p and q are constants.

f. Tasmania knows that it will take him 5 minutes to run the first kilometre and 12.5 minutes to run the first two kilometres.

i. Find the values of p and q.

ii. Find the length of time that Tasmania will take to run the 4 km to reach the buried diamond.

2 + 1 = 3 marks

g. Tasmania takes 19 minutes to dig up the diamond and he is able to run back through the gorge in half the time it took him to reach the diamond. Show that it is possible for him to recover the diamond successfully and state how much time he has to spare.

3 marks

Total 15 marks

Working space

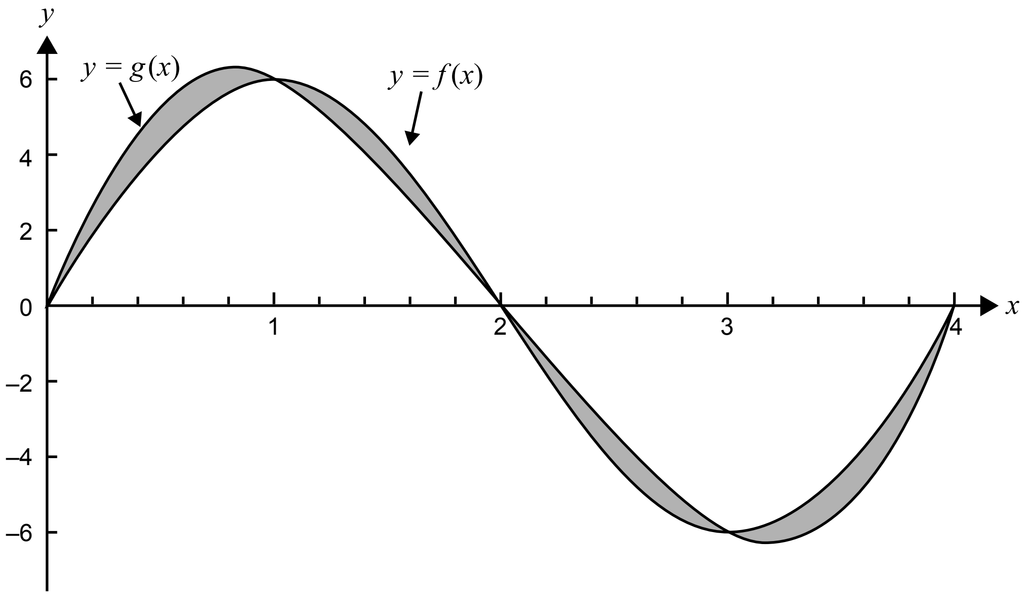
Question 3

Shown below are the graphs of the functions

$$f: [0, 4] \to R, f(x) = 6\sin\left(\frac{\pi x}{2}\right) \text{ and}$$

$$g: [0, 4] \to R, g(x) = 2x(x-2)(x-4) = 2(x^3 - 6x^2 + 8x)$$

The point (1, 6) lies on both graphs.



a. Find the exact value of x for which $g(x)$ is maximum.

2 marks

b. **i.** Write an expression for the total area of the shaded regions using definite integrals.

ii. Find the total area of these shaded regions.

2 + 2 = 4 marks

c. Find, correct to two decimal places, the maximum value of $|f(x) - g(x)|$ for $0 \le x \le 4$ and the values of x in this interval for which this maximum occurs.

2 marks

d. Let $h: [-3, 1] \to R, h(x) = 6\cos\left(\frac{\pi x}{2}\right)$.

i. State a sequence of two transformations which takes the graph of f to the graph of h.

ii. Hence or otherwise, find a cubic polynomial function with domain [–3, 1] that has the same x-intercepts as h and the same maximum and minimum values as g.

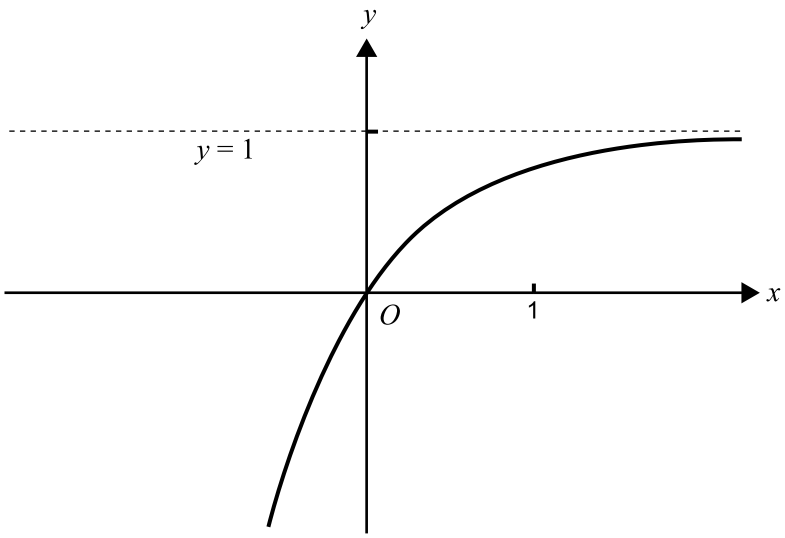
2 + 2 = 4 marks

Total 12 marks

Working space

Question 4

Part of the graph of the function $f: R \to R,\ f(x) = a + be^{-x}$ is shown below. The line with equation $y = 1$ is an asymptote and the graph passes through the origin.



a. Explain why $a = 1$ and $b = -1$.

2 marks

Let h be the function $h: [0, 2] \to R,\ h(x) = f(x)$.

b. State the range of h using exact values.

2 marks

c. **i.** Find the inverse function h^{-1}.

ii. Sketch and label the graph of the inverse function h^{-1} on the axes on page 20.

1 + 2 = 3 marks

d. For $f(x) = 1 - e^{-x}$

i. show that $f(u)\,f(v) = f(u) + f(v) - f(u + v)$, where u and v are real numbers

ii. show that $f(u)\,f(-u) = f(u) + f(-u)$, where u is a real number.

2 + 1 = 3 marks

Total 10 marks

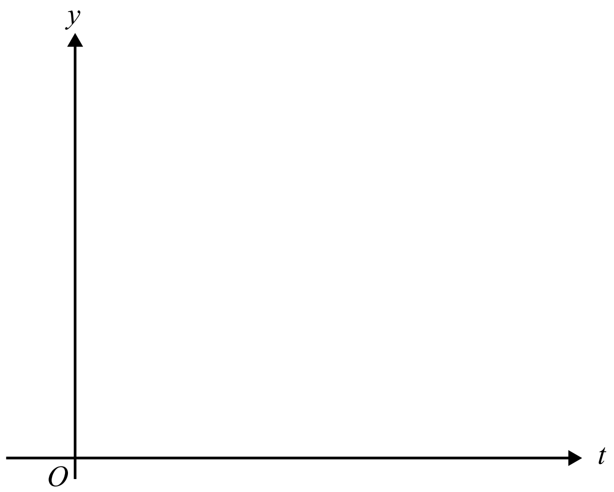
Question 5

In the Great Fun amusement park there is a small train called Puffing Berty which does a circuit of the park.

The continuous random variable T, the time in minutes for a circuit to be completed, has a probability density function f with rule

$$f(t) = \begin{cases} \frac{1}{100}(t-10) & \text{if } 10 \le t < 20 \\ \frac{1}{100}(30-t) & \text{if } 20 \le t \le 30 \\ 0 & \text{otherwise} \end{cases}$$

a. Sketch the graph of $y = f(t)$ on the axes provided.



2 marks

b. Find the probability that the time taken by Puffing Berty to complete a full circuit is less than 25 minutes. (Give the exact value.)

2 marks

c. Find $\Pr(T \le 15 \mid T \le 25)$. (Give the exact value.)

2 marks

The train must complete six circuits between 9.00 am and noon. The management prefers Puffing Berty to complete a circuit in less than 25 minutes.

d. Find the probability, correct to four decimal places, that of the 6 circuits completed, at least 4 of them take less than 25 minutes each.

2 marks

For scheduling reasons the management wants to know the time, b minutes, for which the probability of exactly 3 or 4 out of the 6 circuits completed each taking less than b minutes, is maximised.

Let $\Pr(T < b) = p$
Let Q be the probability that exactly 3 or 4 circuits completed each take less than b minutes.

e. Show that $Q = 5p^3(1-p)^2(4-p)$.

2 marks

f. **i.** Find the maximum value of Q and the value of p for which this occurs. (Give the exact value.)

ii. Find, correct to one decimal place, the value of b for which this maximum occurs.

2 + 2 = 4 marks

Total 14 marks

END OF QUESTION AND ANSWER BOOK

MATHEMATICAL METHODS AND MATHEMATICAL METHODS (CAS)

Written examinations 1 and 2

FORMULA SHEET

Directions to students

Detach this formula sheet during reading time. This formula sheet is provided for your reference.

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Mathematical Methods and Mathematical Methods CAS Formulas

Mensuration

area of a trapezium:	$\frac{1}{2}(a+b)h$	volume of a pyramid:	$\frac{1}{3}Ah$
curved surface area of a cylinder:	$2\pi rh$	volume of a sphere:	$\frac{4}{3}\pi r^3$
volume of a cylinder:	$\pi r^2 h$	area of a triangle:	$\frac{1}{2}bc\sin A$
volume of a cone:	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$

$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$

$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$

$\frac{d}{dx}\left(\sin(ax)\right) = a\ \cos(ax)$

$\frac{d}{dx}\left(\cos(ax)\right) = -a\ \sin(ax)$

$\frac{d}{dx}\left(\tan(ax)\right) = \frac{a}{\cos^2(ax)} = a\ \sec^2(ax)$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, n \neq -1$

$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$\int \frac{1}{x} dx = \log_e |x| + c$

$\int \sin(ax) dx = -\frac{1}{a}\cos(ax) + c$

$\int \cos(ax) dx = \frac{1}{a}\sin(ax) + c$

product rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

chain rule: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

approximation: $f(x+h) \approx f(x) + hf'(x)$

Probability

$\Pr(A) = 1 - \Pr(A')$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$

mean: $\mu = \mathrm{E}(X)$

variance: $\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X-\mu)^2) = \mathrm{E}(X^2) - \mu^2$

probability distribution		mean	variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x-\mu)^2\, p(x)$
continuous	$\Pr(a < X < b) = \int_a^b f(x)dx$	$\mu = \int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

END OF FORMULA SHEET