VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2018

SUPERVISOR TO ATTACH PROCESSING LABEL HERE

Letter

STUDENT NUMBER

SPECIALIST MATHEMATICS

Written examination 1

Friday 9 November 2018

Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	*Number of questions to be answered*	*Number of marks*
10	10	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 9 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

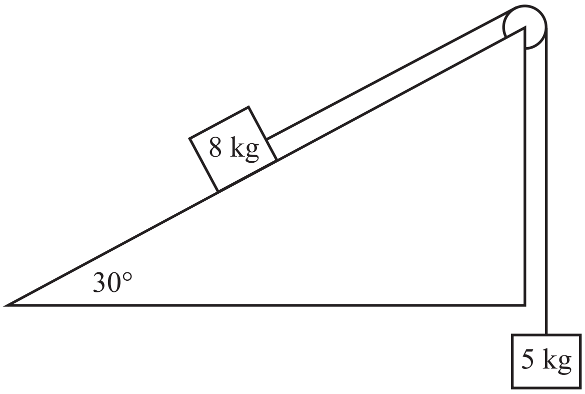
THIS PAGE IS BLANK

Instructions

Answer **all** questions in the spaces provided.
Unless otherwise specified, an **exact** answer is required to a question.
In questions where more than one mark is available, appropriate working **must** be shown.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g ms^{-2}, where $g = 9.8$

Question 1 (4 marks)

Two objects of masses 5 kg and 8 kg are attached by a light inextensible string that passes over a smooth pulley. The 8 kg mass is on a smooth plane inclined at 30° to the horizontal. The 5 kg mass is hanging vertically, as shown in the diagram below.



a. On the diagram above, show all forces acting on both masses. 1 mark

b. Find the magnitude, in ms^{-2}, and state the direction of the acceleration of the 8 kg mass. 3 marks

Question 2 (4 marks)

a. Show that $1+i=\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$. 1 mark

b. Evaluate $\frac{\left(\sqrt{3}-i\right)^{10}}{(1+i)^{12}}$, giving your answer in the form $a + bi$, where $a, b \in R$. 3 marks

Question 3 (4 marks)

Find the gradient of the curve with equation $2x^2\sin(y)+xy=\frac{\pi^2}{18}$ at the point $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Give your answer in the form $\frac{a}{\pi\sqrt{b}+c}$, where a, b and c are integers.

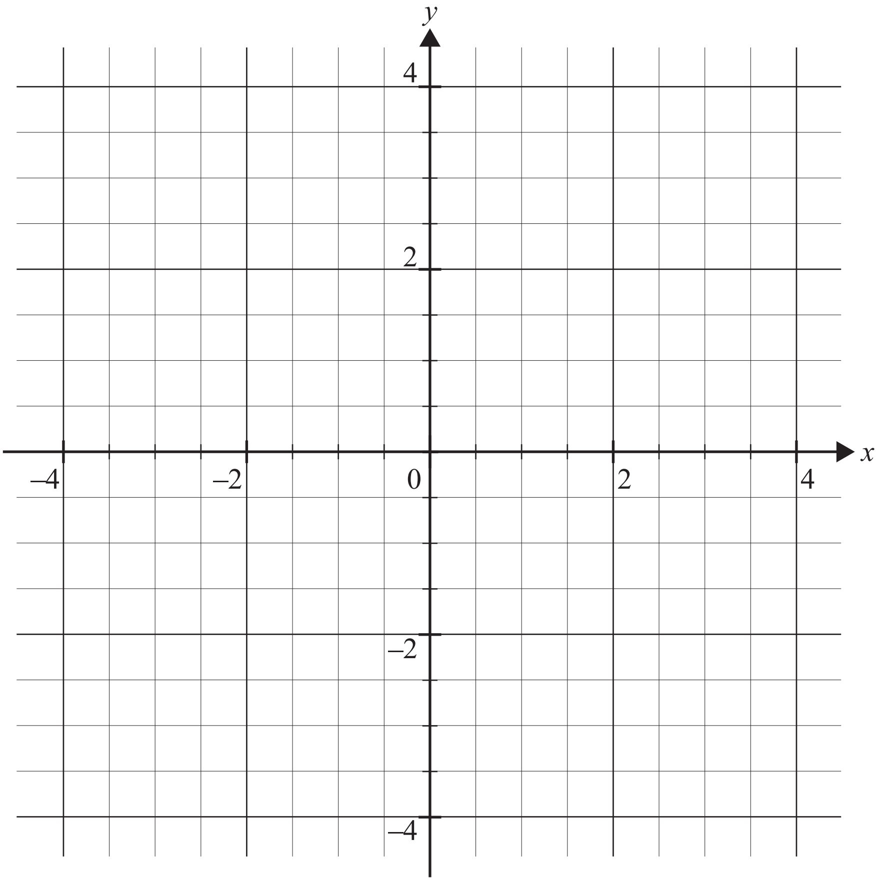
Question 4 (4 marks)

X and Y are independent random variables. The mean and the variance of X are both 2, while the mean and the variance of Y are 2 and 4 respectively.

Given that a and b are integers, find the values of a and b if the mean and the variance of $aX + bY$ are 10 and 44 respectively.

Question 5 (4 marks)

Sketch the graph of $f(x) = \dfrac{x+1}{x^2-4}$ on the axes provided below, labelling any asymptotes with their equations and any intercepts with their coordinates.



Question 6 (3 marks)

A particle of mass 2 kg moves under a force $\underset{\sim}{F}$ so that its position vector $\underset{\sim}{r}$ at any time t is given by $\underset{\sim}{r} = \sin(t)\,\underset{\sim}{i} + \cos(t)\,\underset{\sim}{j} + t^2\,\underset{\sim}{k}$. Distances are measured in metres and time is measured in seconds.

Find the change in momentum, in kg ms^{-2}, from $t = \dfrac{\pi}{2}$ to $t = \pi$.

Question 7 (3 marks)

Given that $\cot(2x) + \dfrac{1}{2}\tan(x) = a\cot(x)$, use a suitable double angle formula to find the value of a, $a \in R$.

Question 8 (4 marks)

A tank initially holds 16 L of water in which 0.5 kg of salt has been dissolved. Pure water then flows into the tank at a rate of 5 L per minute. The mixture is stirred continuously and flows out of the tank at a rate of 3 L per minute.

a. Show that the differential equation for Q, the number of kilograms of salt in the tank after t minutes, is given by 1 mark

$$\frac{dQ}{dt} = -\frac{3Q}{16+2t}$$

b. Solve the differential equation given in **part a.** to find Q as a function of t. Express your answer in the form $Q = \frac{a}{(16+2t)^{\frac{b}{c}}}$, where a, b and c are positive integers. 3 marks

Question 9 (5 marks)

A curve is specified parametrically by $\underset{\sim}{r}(t) = \sec(t)\,\underset{\sim}{i} + \frac{\sqrt{2}}{2}\tan(t)\,\underset{\sim}{j},\ t \in R$.

a. Show that the cartesian equation of the curve is $x^2 - 2y^2 = 1$. 2 marks

b. Find the x-coordinates of the points of intersection of the curve $x^2 - 2y^2 = 1$ and the line $y = x - 1$. 1 mark

c. Find the volume of the solid of revolution formed when the region bounded by the curve and the line is rotated about the x-axis. 2 marks

Question 10 (5 marks)

The position vector of a particle moving along a curve at time t seconds is given by

$\underset{\sim}{r}(t) = \frac{t^3}{3}\underset{\sim}{i} + \left(\arcsin(t) + t\sqrt{1-t^2}\right)\underset{\sim}{j}$, $0 \le t \le 1$, where distances are measured in metres.

The distance d metres that the particle travels along the curve in three-quarters of a second is given by

$$d = \int_0^{\frac{3}{4}} \left(at^2 + bt + c\right) dt$$

Find a, b and c, where $a, b, c \in Z$.

END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Victorian Certificate of Education 2018

SPECIALIST MATHEMATICS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference. A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Specialist Mathematics formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$
curved surface area of a cylinder	$2\pi rh$
volume of a cylinder	$\pi r^2 h$
volume of a cone	$\frac{1}{3}\pi r^2 h$
volume of a pyramid	$\frac{1}{3}Ah$
volume of a sphere	$\frac{4}{3}\pi r^3$
area of a triangle	$\frac{1}{2}bc\sin(A)$
sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$	$\tan(x-y) = \frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	
$\sin(2x) = 2\sin(x)\cos(x)$	$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)}$

Circular functions – continued

Function	$\sin^{-1}$ or arcsin	$\cos^{-1}$ or arccos	$\tan^{-1}$ or arctan
Domain	$[-1, 1]$	$[-1, 1]$	R
Range	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$[0, \pi]$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Algebra (complex numbers)

$z = x + iy = r\left(\cos(\theta) + i\sin(\theta)\right) = r\,\mathrm{cis}(\theta)$	
$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	$-\pi < \mathrm{Arg}(z) \le \pi$
$z_1 z_2 = r_1 r_2\,\mathrm{cis}(\theta_1 + \theta_2)$	$\frac{z_1}{z_2} = \frac{r_1}{r_2}\mathrm{cis}\left(\theta_1 - \theta_2\right)$
$z^n = r^n\,\mathrm{cis}(n\theta)$ (de Moivre's theorem)	

Probability and statistics

for random variables X and Y	$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$ $\mathrm{E}(aX + bY) = a\mathrm{E}(X) + b\mathrm{E}(Y)$ $\mathrm{var}(aX + b) = a^2\mathrm{var}(X)$
for independent random variables X and Y	$\mathrm{var}(aX + bY) = a^2\mathrm{var}(X) + b^2\mathrm{var}(Y)$
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}},\ \bar{x} + z\frac{s}{\sqrt{n}}\right)$
distribution of sample mean $\bar{X}$	mean $\mathrm{E}\left(\bar{X}\right) = \mu$ variance $\mathrm{var}\left(\bar{X}\right) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$
$\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$
$\frac{d}{dx}\left(\log_e(x)\right) = \frac{1}{x}$	$\int \frac{1}{x}\,dx = \log_e\lvert x\rvert + c$
$\frac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\frac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\frac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$	$\int \frac{a}{a^2+x^2}\,dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
	$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c,\ n \neq -1$
	$\int (ax+b)^{-1}\,dx = \frac{1}{a}\log_e\lvert ax+b\rvert + c$
product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
Euler's method	If $\frac{dy}{dx} = f(x)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + hf(x_n)$
acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$
arc length	$\int_{x_1}^{x_2}\sqrt{1+\left(f'(x)\right)^2}\,dx$ or $\int_{t_1}^{t_2}\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2}\,dt$

Vectors in two and three dimensions

$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$
$\lvert \underset{\sim}{r} \rvert = \sqrt{x^2+y^2+z^2} = r$
$\dot{\underset{\sim}{r}} = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
$\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = r_1 r_2\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$

Mechanics

momentum	$\underset{\sim}{p} = m\underset{\sim}{v}$
equation of motion	$\underset{\sim}{R} = m\underset{\sim}{a}$

END OF FORMULA SHEET