Letter

STUDENT NUMBER

MATHEMATICAL METHODS
Written examination 1

Wednesday 1 November 2023

**Reading time: 9.00 am to 9.15 am (15 minutes)
Writing time: 9.15 am to 10.15 am (1 hour)**

QUESTION AND ANSWER BOOK

Structure of book

Number of questions	Number of questions to be answered	Number of marks
9	9	40

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners and rulers.
- Students are NOT permitted to bring into the examination room: any technology (calculators or software), notes of any kind, blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 14 pages
- Formula sheet
- Working space is provided throughout the book.

Instructions
- Write your **student number** in the space provided above on this page.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

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Instructions

Answer **all** questions in the spaces provided.

In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

Question 1 (4 marks)

a. Let $y = \dfrac{x^2 - x}{e^x}$.

Find and simplify $\dfrac{dy}{dx}$.   2 marks

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b. Let $f(x) = \sin(x)e^{2x}$.

Find $f'\left(\dfrac{\pi}{4}\right)$.   2 marks

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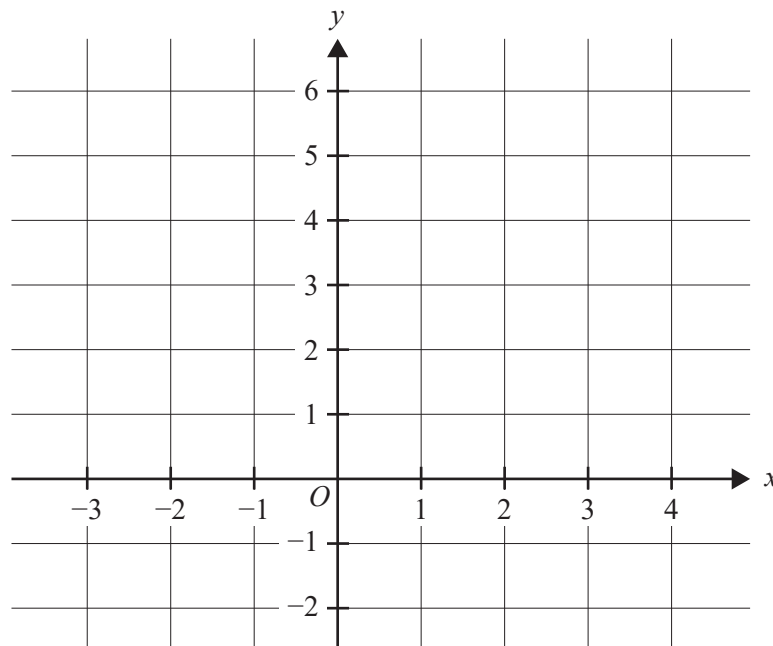
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Question 2 (3 marks)

Solve $e^{2x} - 12 = 4e^x$ for $x \in R$.

Question 3 (4 marks)

a. Sketch the graph of $f(x) = 2 - \dfrac{3}{x-1}$ on the axes below, labelling all asymptotes with their equations

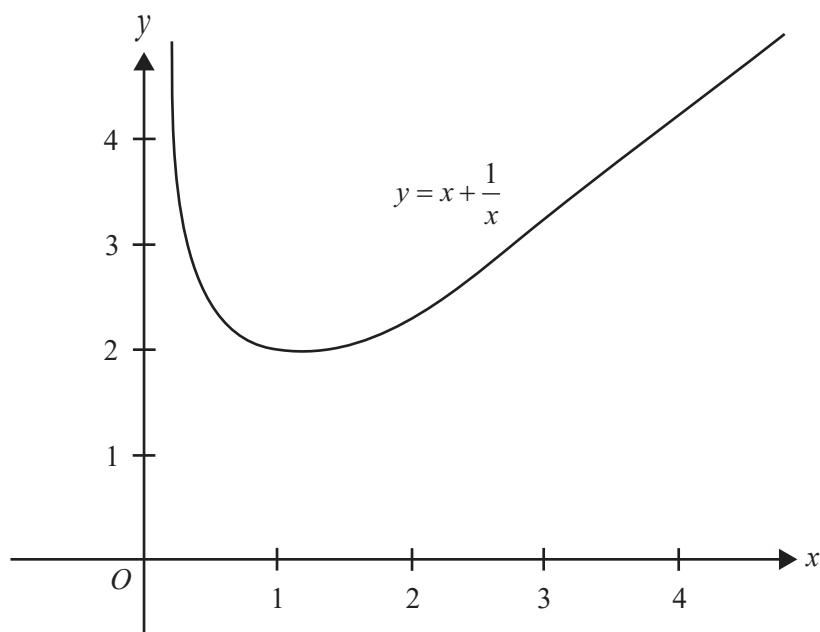and axial intercepts with their coordinates.   3 marks

b. Find the values of x for which $f(x) \leq 1$.   1 mark

Question 4 (2 marks)

The graph of $y = x + \dfrac{1}{x}$ is shown over part of its domain.

Use two trapeziums of equal width to approximate the area between the curve, the x-axis and the lines $x = 1$ and $x = 3$.

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Question 5 (4 marks)

a. Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\,dx$.   1 mark

b. Hence, or otherwise, find all values of k such that $\displaystyle\int_0^{\frac{\pi}{3}} \sin(x)\,dx = \int_k^{\frac{\pi}{2}} \cos(x)\,dx$, where $-3\pi < k < 2\pi$. 3 marks

Question 6 (4 marks)

Let $\hat{P}$ be the random variable that represents the sample proportion of households in a given suburb that have solar panels installed.

From a sample of randomly selected households in a given suburb, an approximate 95% confidence interval for the proportion p of households having solar panels installed was determined to be (0.04, 0.16).

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

Use $z = 2$ to approximate the 95% confidence interval.

b. Find the size of the sample from which this 95% confidence interval was obtained. 2 marks

c. A larger sample of households is selected, with a sample size four times the original sample. The sample proportion of households having solar panels installed is found to be the same.
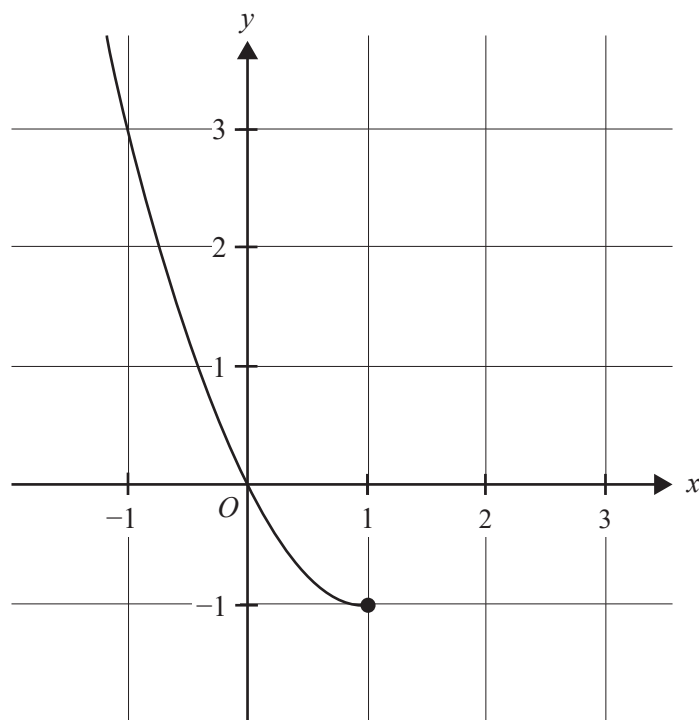
By what factor will the increased sample size affect the width of the confidence interval? 1 mark

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Question 7 (7 marks)

Consider $f: (-\infty, 1] \to R, f(x) = x^2 - 2x$. Part of the graph of $y = f(x)$ is shown below.

a.   State the range of f.   1 mark

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b.   Sketch the graph of the inverse function $y = f^{-1}(x)$ on the axes above. Label any endpoints and axial intercepts with their coordinates.   2 marks

c.   Determine the equation and the domain for the inverse function f^{-1}.   2 marks

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d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$.   2 marks

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Question 8 (6 marks)

Suppose that the queuing time, T (in minutes), at a customer service desk has a probability density function given by

$$f(t) = \begin{cases} kt(16 - t^2) & 0 \le t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

for some $k \in R$.

a.   Show that $k = \dfrac{1}{64}$.   1 mark

b.   Find $E(T)$.   2 marks

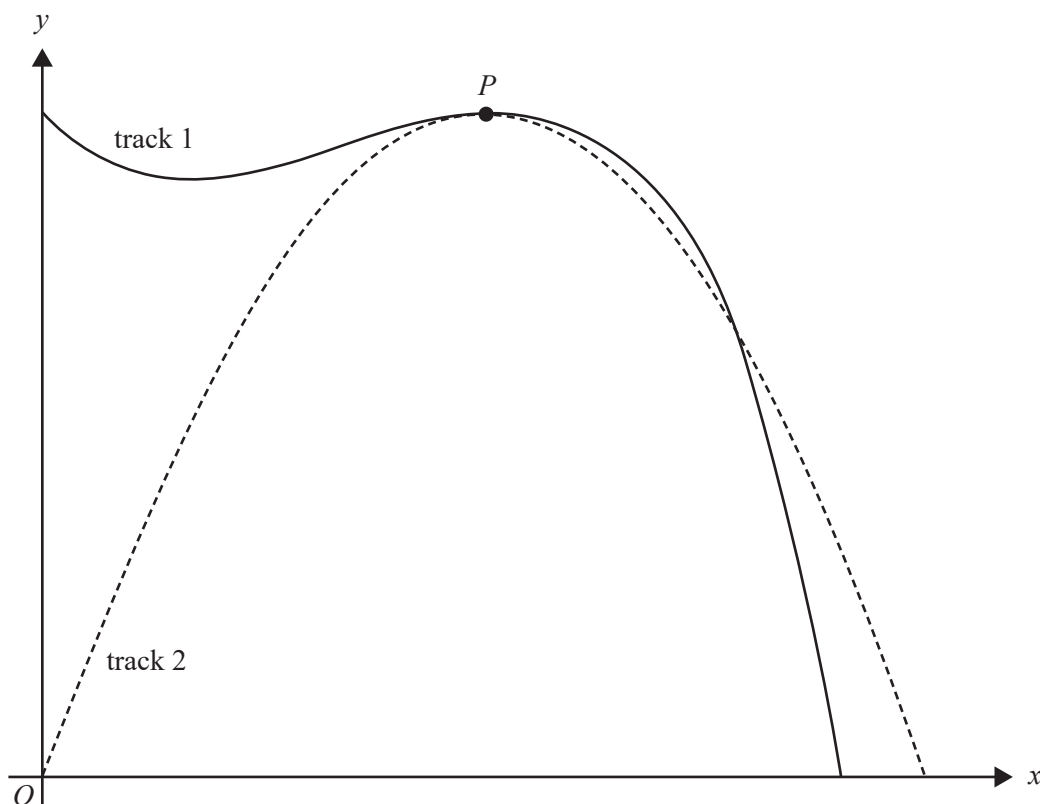**c.** What is the probability that a person has to queue for more than two minutes, given that they have already queued for one minute? 3 marks

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Question 9 (6 marks)

The shapes of two walking tracks are shown below.

Track 1 is described by the function $f(x) = a - x(x - 2)^2$.

Track 2 is defined by the function $g(x) = 12x + bx^2$.

The unit of length is kilometres.

a.   Given that $f(0) = 12$ and $g(1) = 9$, verify that $a = 12$ and $b = -3$.   1 mark

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b.   Verify that $f(x)$ and $g(x)$ both have a turning point at P.
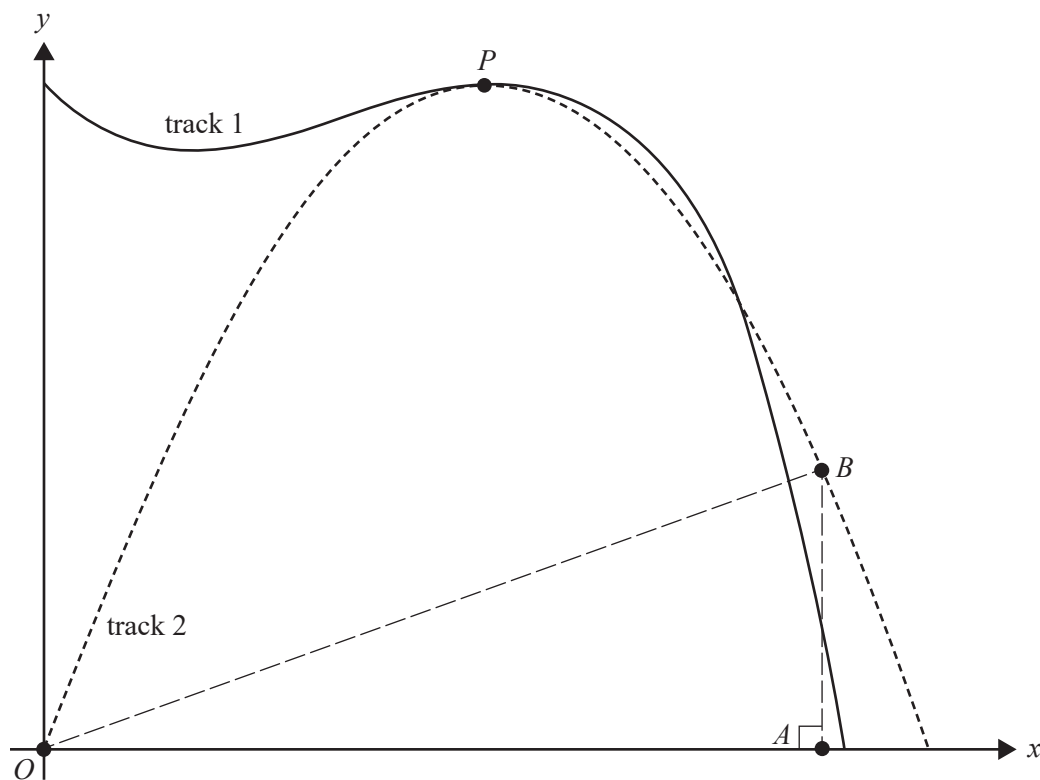Give the co-ordinates of P.   2 marks

c. A theme park is planned whose boundaries will form the triangle ΔOAB where O is the origin, A is at $(k, 0)$ and B is at $\left(k, g(k)\right)$, as shown below, where $k \in (0, 4)$.

Find the maximum possible area of the theme park, in km^2.   3 marks

**Victorian Certificate of Education
2023**

MATHEMATICAL METHODS

Written examination 1

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

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Mathematical Methods formulas

Mensuration

area of a trapezium	$\frac{1}{2}(a+b)h$	volume of a pyramid	$\frac{1}{3}Ah$
curved surface area of a cylinder	$2\pi rh$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$		

Calculus

$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$	$\int x^n dx = \frac{1}{n+1}x^{n+1}+c,\ n\neq-1$
$\frac{d}{dx}\left((ax+b)^n\right)=an(ax+b)^{n-1}$	$\int (ax+b)^n dx = \frac{1}{a(n+1)}(ax+b)^{n+1}+c, n\neq-1$
$\frac{d}{dx}\left(e^{ax}\right)=ae^{ax}$	$\int e^{ax} dx = \frac{1}{a}e^{ax}+c$
$\frac{d}{dx}\left(\log_e(x)\right)=\frac{1}{x}$	$\int \frac{1}{x}dx = \log_e(x)+c,\ x>0$
$\frac{d}{dx}\left(\sin(ax)\right)=a\cos(ax)$	$\int \sin(ax)dx = -\frac{1}{a}\cos(ax)+c$
$\frac{d}{dx}\left(\cos(ax)\right)=-a\sin(ax)$	$\int \cos(ax)dx = \frac{1}{a}\sin(ax)+c$
$\frac{d}{dx}\left(\tan(ax)\right)=\frac{a}{\cos^2(ax)}=a\sec^2(ax)$	

product rule	$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$	quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$	Newton's method	$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$
trapezium rule approximation	$Area \approx \frac{x_n-x_0}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-2})+2f(x_{n-1})+f(x_n)\right]$		

Probability

$\Pr(A) = 1 - \Pr(A')$	$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
$\Pr(A\|B) = \dfrac{\Pr\left(A \cap B\right)}{\Pr\left(B\right)}$	

mean	$\mu = \mathrm{E}(X)$	variance	$\mathrm{var}(X) = \sigma^2 = \mathrm{E}((X - \mu)^2) = \mathrm{E}(X^2) - \mu^2$
binomial coefficient	$\dbinom{n}{x} = \dfrac{n!}{x!(n-x)!}$		

Probability distribution		Mean	Variance
discrete	$\Pr(X = x) = p(x)$	$\mu = \sum x\, p(x)$	$\sigma^2 = \sum (x - \mu)^2\, p(x)$
binomial	$\Pr(X = x) = \dbinom{n}{x} p^x (1-p)^{n-x}$	$\mu = np$	$\sigma^2 = np(1-p)$
continuous	$\Pr(a < X < b) = \displaystyle\int_a^b f(x)dx$	$\mu = \displaystyle\int_{-\infty}^{\infty} x\, f(x)dx$	$\sigma^2 = \displaystyle\int_{-\infty}^{\infty} (x - \mu)^2\, f(x)dx$

Sample proportions

$\hat{P} = \dfrac{X}{n}$		mean	$\mathrm{E}(\hat{P}) = p$
standard deviation	$\mathrm{sd}(\hat{P}) = \sqrt{\dfrac{p(1-p)}{n}}$	approximate confidence interval	$\left(\hat{p} - z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}},\ \hat{p} + z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} \right)$

END OF FORMULA SHEET