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Write your **student number** in the boxes above. **Letter**

Specialist Mathematics Examination 1

Question and Answer Book

VCE (NHT) Examination – Thursday 23 May 2024

- Reading time is **15 minutes**: 10.30 am to 10.45 am
- Writing time is **1 hour**: 10.45 am to 11.45 am

Materials supplied

- Question and Answer Book of 12 pages
- Formula Sheet

Students are **not** permitted to bring any technology (calculators or software) or notes of any kind into the examination room.

Students are **not** permitted to bring mobile phones and/or any unauthorised electronic devices into the examination room.

Contents

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Instructions

- Answer **all** questions in the spaces provided.
- Write your responses in English.
- Unless otherwise specified, an **exact** answer is required for each question.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- Take the **acceleration due to gravity** to have a magnitude $g \text{ m s}^{-2}$, where $g = 9.8$

Question 1 (4 marks)

a. A certain model of electric car accelerates in a straight line from rest, with constant acceleration, to travel 400 metres in 10 seconds.

i. Find the acceleration of the car in m s^{-2}. 2 marks

ii. Find the speed of the car, in m s^{-1}, when it has travelled 400 m. 1 mark

b. A second car accelerates in a straight line from rest, with constant acceleration, for 12 seconds.

Given that the speed of this car is $V \text{ m s}^{-1}$ after 12 seconds, write down the average speed of the car in terms of V for the first 6 seconds of its motion. 1 mark

Question 2 (3 marks)

Solve $z^4 = -8 + 8\sqrt{3}i$, giving your answers in Cartesian form.

Question 3 (3 marks)

a. Find the coordinates of the points of inflection of the graph of $y = x^4 - 6x^2 + 4$. 2 marks

b. Hence, or otherwise, find all values of x for which the graph of $y = x^4 - 6x^2 + 4$ is concave up. 1 mark

Question 4 (3 marks)

Find the coordinates of the point where the line given by the equation $\underset{\sim}{r} = 4\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k} + t(2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k})$, where $t \in R$, intersects the plane with the equation $2x + 3y - z = 6$.

Question 5 (5 marks)

The curve defined by $6e^{y} \arctan\left(\frac{x}{\sqrt{3}}\right) - \pi \cos(y) = 0$ has an x-intercept of 1.

Find the equation of the straight line that is **perpendicular** to the curve at this intercept.

Give your answer in the form $y = mx + c$, where $m, c \in R$.

Question 6 (4 marks)

a. A café made a large batch of muffins. The masses of the muffins are normally distributed with a mean of μ grams and a standard deviation of 1 gram.

A random sample of 16 muffins was selected from the batch, and the sample mean was 25 grams.

Find a 95% confidence interval $(z = 1.96)$ for μ. Give your answer correct to one decimal place. 2 marks

b. A different random sample of 25 muffins was selected from the batch. The mean mass of this new sample was 25 grams, and a 95% confidence interval was calculated correct to one decimal place.

By what percentage is the width of the new confidence interval reduced compared to the confidence interval calculated from the sample in **part a**? 2 marks

Question 7 (4 marks)

The curve defined by the parametric equations $x = 8t$ and $y = t^2 - 8\log_e(t)$, where $t \in [1, 3]$, is rotated about the y-axis to form a surface of revolution.

Find the area of the surface of revolution formed.

Give your answer in the form $\frac{a\pi}{b}$ where $a, b \in Z^+$.

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Do not write in this area.

Question 8 (4 marks)

Prove by mathematical induction that

$1 \times 7 + 2 \times 15 + 3 \times 23 + \ \dots \ + n(8n-1) = \frac{1}{6}n(n+1)(16n+5)$ for all $n \in N$.

Question 9 (6 marks)

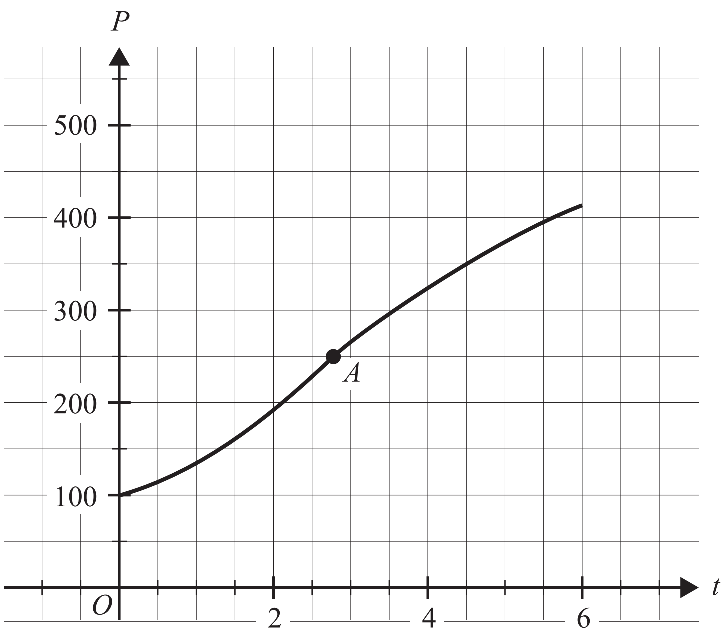
In a population, the number of kangaroos, P, is modelled by the logistic differential equation

$\frac{dP}{dt} = 0.05P\left(10 - \frac{P}{50}\right)$, where $P \in [100, 500)$ and where t is the time in years after the population was established.

The initial number of kangaroos in the population was 100.

a. Find $\frac{d^2P}{dt^2}$ in terms of P. 1 mark

Part of the graph of P against t is shown below. The graph contains a single point of inflection, labelled A.



b. In the context of the growth of the kangaroo population, what does the point of inflection at A represent? 1 mark

c. Solve the logistic differential equation to show that $t = 2\log_e\left(\frac{4P}{500 - P}\right)$. 3 marks

d. Hence, or otherwise, find the coordinates of point A. 1 mark

Question 10 (4 marks)

Use integration by parts to evaluate $\int_{\frac{3}{2}}^{\frac{3\sqrt{3}}{2}} \left(4\arcsin\left(\frac{x}{3}\right)\right) dx$.

Give your answer in the form $a\pi + b$, where $a, b \in R$.

End of examination questions

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Specialist Mathematics Examination 1

Formula Sheet

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\operatorname{cis}(\theta)$	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \operatorname{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \operatorname{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\mathrm{E}(aX_1 + b) = a\,\mathrm{E}(X_1) + b$ $\mathrm{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\mathrm{E}(X_1) + a_2\mathrm{E}(X_2) + \ldots + a_n\mathrm{E}(X_n)$
	$\mathrm{Var}(aX_1 + b) = a^2\mathrm{Var}(X_1)$ $\mathrm{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\mathrm{Var}(X_1) + a_2^2\mathrm{Var}(X_2) + \ldots + a_n^2\mathrm{Var}(X_n)$
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = n\mu$
	$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$

You may keep this Formula Sheet.

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Data analysis, probability and statistics – continued

approximate confidence interval for μ	$\left(\bar{x} - z\dfrac{s}{\sqrt{n}},\ \bar{x} + z\dfrac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\mathrm{E}\left(\bar{X}\right) = \mu$
	variance	$\mathrm{Var}\left(\bar{X}\right) = \dfrac{\sigma^2}{n}$

Calculus

$\dfrac{d}{dx}\left(x^n\right) = n x^{n-1}$	$\int x^n dx = \dfrac{1}{n+1} x^{n+1} + c,\quad n \neq -1$
$\dfrac{d}{dx}\left(e^{ax}\right) = a e^{ax}$	$\int e^{ax} dx = \dfrac{1}{a} e^{ax} + c$
$\dfrac{d}{dx}\left(\log_e(x)\right) = \dfrac{1}{x}$	$\int \dfrac{1}{x} dx = \log_e \lvert x \rvert + c$
$\dfrac{d}{dx}\left(\sin(ax)\right) = a\cos(ax)$	$\int \sin(ax)\,dx = -\dfrac{1}{a}\cos(ax) + c$
$\dfrac{d}{dx}\left(\cos(ax)\right) = -a\sin(ax)$	$\int \cos(ax)\,dx = \dfrac{1}{a}\sin(ax) + c$
$\dfrac{d}{dx}\left(\tan(ax)\right) = a\sec^2(ax)$	$\int \sec^2(ax)\,dx = \dfrac{1}{a}\tan(ax) + c$
$\dfrac{d}{dx}\left(\cot(ax)\right) = -a\,\mathrm{cosec}^2(ax)$	$\int \mathrm{cosec}^2(ax)dx = -\dfrac{1}{a}\cot(ax) + c$
$\dfrac{d}{dx}\left(\sec(ax)\right) = a\sec(ax)\tan(ax)$	$\int \sec(ax)\tan(ax)\,dx = \dfrac{1}{a}\sec(ax) + c$
$\dfrac{d}{dx}\left(\mathrm{cosec}(ax)\right) = -a\,\mathrm{cosec}(ax)\cot(ax)$	$\int \mathrm{cosec}(ax)\cot(ax)\,dx = -\dfrac{1}{a}\mathrm{cosec}(ax) + c$
$\dfrac{d}{dx}\left(\sin^{-1}(ax)\right) = \dfrac{a}{\sqrt{1-(ax)^2}}$	$\int \dfrac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c,\ a > 0$
$\dfrac{d}{dx}\left(\cos^{-1}(ax)\right) = \dfrac{-a}{\sqrt{1-(ax)^2}}$	$\int \dfrac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\dfrac{x}{a}\right) + c,\ a > 0$
$\dfrac{d}{dx}\left(\tan^{-1}(ax)\right) = \dfrac{a}{1+(ax)^2}$	$\int \dfrac{a}{a^2 + x^2} dx = \tan^{-1}\left(\dfrac{x}{a}\right) + c$
	$\int (ax+b)^n\,dx = \dfrac{1}{a(n+1)}(ax+b)^{n+1} + c,\quad n \neq -1$
	$\int \dfrac{1}{ax+b} dx = \dfrac{1}{a}\log_e \lvert ax+b \rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
Euler's method	If $\frac{dy}{dx} = f(x, y)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$.
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u + v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \qquad y(t) = y_1 + y_2t \qquad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s, t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s, t) = x_0 + x_1s + x_2t,\ y(s, t) = y_0 + y_1s + y_2t,\ z(s, t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

End of Formula Sheet