VCE Specialist Mathematics

Written examination 1 – End of year

Sample questions

These sample questions are intended to demonstrate how new aspects of Units 3 and 4 of VCE Specialist Mathematics written examination 1 may be examined. They do **not** constitute a full examination paper.

Question 1 (4 marks)

Consider the statement $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$, where $n \in N$.

a. Show that if $n = 1$, the statement is true. 1 mark

b. Assume that the statement is true for $n = k$.

Write down the assumption in terms of k. 1 mark

c. Hence, prove by mathematical induction that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+...+\frac{1}{2^n}=1-\frac{1}{2^n}$, where $n \in N$. 2 marks

Question 2 (4 marks)

a. Consider the inequality $2^n > n^2$ for $n \geq n_0$, where $n \in N$.

Show that $n_0 = 5$. 1 mark

b. Prove by mathematical induction that $2^n > n^2$ for $n \geq 5$, where $n \in N$. 3 marks

Question 3 (4 marks)

Prove by mathematical induction that the number $9^n - 5^n$ is divisible by 4 for all $n \in N$.

Question 4 (3 marks)

Use proof by contradiction to prove that if n is odd, where $n \in N$, then $n^3 + 1$ is even.

Question 5 (3 marks)

Use proof by contradiction to prove that $\sqrt{3}+\sqrt{5}>\sqrt{11}$.

Question 6 (4 marks)

The curve given by $y=\sqrt{4-x^2}$, where $x \in [-1, 1]$, is rotated about the x-axis to form a solid of revolution.

Find the surface area of this solid of revolution.

Question 7 (5 marks)

The curve given by $y = \sqrt[3]{x}$ is rotated about the y-axis to form a solid of revolution.

Find the surface area of the part of this solid of revolution where $x \in [0, 8]$.

Question 8 (4 marks)

Determine the surface area obtained by rotating the curve defined by the parametric equations $x = \sin^3(\theta)$, $y = \cos^3(\theta)$, where $\theta \in \left[0, \frac{\pi}{2}\right]$, about the y-axis.

Question 9 (3 marks)

Find the surface area of revolution formed when the curve defined by the parametric equations $x = \frac{4}{3}\sqrt{(t+1)^3}$, $y = \frac{1}{2}t^2$, where $0 \le t \le 1$, is rotated about the x-axis.

Question 10 (7 marks)

The population of bacteria, $P(t)$, in a Petri dish satisfies the logistic differential equation

$$\frac{dP}{dt} = 2P\left(6 - \frac{P}{8000}\right)$$

where t is measured in hours and the initial population is 4000 bacteria.

a. Find the maximum number of bacteria predicted by this model. 1 mark

b. Find the number of bacteria when the population is growing at its fastest rate. 2 marks

c. Solve the differential equation to find P as a function of t. 4 marks

Question 11 (4 marks)

Find $\int x^2 \cos(2x)\, dx$.

Question 12 (3 marks)

The vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$ lie in a plane that passes through the point (3, 2, 1).

Find the Cartesian equation of this plane.

Question 13 (6 marks)

a. Find the equation of the plane that passes through the points $P(3, 3, 6)$, $Q(1, -1, 2)$ and $R(5, 2, 0)$. 4 marks

b. Find the point of intersection of the line given by $\underset{\sim}{r} = 2\underset{\sim}{i} + 5\underset{\sim}{k} + t(2\underset{\sim}{i} - 4\underset{\sim}{j} - 3\underset{\sim}{k})$, where $t \in R$, with the plane given by $2x - 2y + z = 6$. 2 marks

Question 14 (3 marks)

Find the angle between the plane given by $2x + y + z = 7$ and the line given by $\underset{\sim}{r} = 11\underset{\sim}{i} + 4\underset{\sim}{j} + 3\underset{\sim}{k} + t(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k})$, where $t \in R$.

Question 15 (5 marks)

a. Find the vector equation of the line through the points $A(3, 1, -1)$ and $B(5, 2, -6)$. 2 marks

b. Find the sine of the angle that this line makes with the plane given by $x + 2y - z = 9$. 3 marks

Question 16 (4 marks)

The position of a particle after t seconds is given by $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + 5t\,\underset{\sim}{j} + (t^2 - 16t)\underset{\sim}{k}$, where $t \geq 0$ and components are measured in metres.

Find the time at which the minimum speed occurs and calculate the minimum speed. Give your answer in m s^{-1}.

Question 17 (3 marks)

Two planes have equations $x + y - z = 3$ and $2x - y - 2z = 4$.

Given that the angle between the two planes is θ, find $\sec(\theta)$.

Question 18 (3 marks)

The position vectors $\underset{\sim}{a} = 2\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ form two sides of a triangle.

Find the area of the triangle in the form $c\sqrt{d}$, where $c, d \in N$.

Question 19 (4 marks)

A parallelogram, *OABC*, has vertices at $O(0, 0, 0)$, $A(1, 2, -1)$ and $C(3, m, 1)$, where $m \in R$.

Find the value(s) of m if the area of the parallelogram is $4\sqrt{5}$.