**Paper 3: Deeper review – directions and options**

Introduction

This paper provides further background and context for a deeper review of senior secondary mathematics. It is the third in a series of papers developed for this work and includes meta-analysis of a range of invited responses to the question:

*What could a suitable senior secondary mathematics curriculum for a liberal democratic society in a developed country for 2020–2030 look like*?

Respondents were encouraged to be ‘blue-sky’ – not limited by current thinking or beliefs – and ‘green-field’ – not constrained by prior work, an opportunity to start from scratch in their considerations. This paper is structured as follows:

1. Background
2. Societal context
3. The nature of the review work
4. Mathematics
5. Mathematics education
6. Meta-analysis of views, ideas and approaches

1. Background

At its October 2017 meeting the VCAA Senior Secondary Curriculum and Assessment Committee (SSCAC) provided direction for further work in 2018 in preparation for an in-depth review of the senior secondary mathematics curriculum. The decision was informed by:

* the paper *Senior Secondary Mathematics in Victoria: Working Towards Change,* a historical overview of developments in senior secondary mathematics curriculum in Victoria from 1980–2016
* the discussion paper *New Directions* (September 2014)
* developments in the discipline, in mathematics education and in society over the past several decades, and
* consideration of these papers and developments in the light of the most recent review following on from the Australian Curriculum (AC) senior secondary curriculum work 2010–2012.

The current VCE Mathematics study design has evolved from a study design first developed in 1993, and subsequently refined through five successive cycles of review, accreditation and implementation. The VCE Mathematics study structure is like that of the AC senior secondary curriculum developed during the work undertaken nationally in 2010–12.

The AC structure substantially informed the subsequent cycle of senior secondary review in jurisdictions around Australia, with implementation of revised courses in the ACT, South Australia/Northern Territory, Tasmania, Victoria, Western Australia and, more recently, Queensland (Year 11 in 2019, introducing external examinations from 2020) and New South Wales (introducing examinations from 2019–20, depending on the course).

The key impact of the AC work in Victoria was to support the inclusion of content on *introductory statistical inference* to complement existing content on probability in the calculus-based VCE Mathematic studies. The 2013–14 Victorian VCE Mathematics review also incorporated further emphasis on *computational thinking, algorithms* and *recursive processes*.

The [VCE Mathematics study design](http://www.vcaa.vic.edu.au/Documents/vce/mathematics/MathematicsSD-2016.pdf) has been progressively developed so that it enables a more general and realistic range of problems to be tackled as computational complexity, including algebraic computation, is not limited to the ‘average-best’ by hand capability of students. This is due to the active use of technology which has been an explicit expectation of the study design from 2000. The current VCE Mathematics study design is one of the more advanced in the world in this respect, with a focus on the use of applicable functionality of technology for computation, rather than the nature of the platform or device used, and clear specification of expectations for mental, by hand and technology-based computation.

2. Societal context – now, five years, ten years

The next decade 2020–30 starts in a bit less than a year. If the current decade is any indication, developments in the increasingly globalized world move very quickly, especially where technology is involved. From 1980 to 2015, the percentage of 15 to 24-year-olds employed full time (the so called [quarter-life crisis](https://www.theage.com.au/national/victoria/25-and-unemployed-the-quarter-life-crisis-hurting-young-people-20180613-p4zl9s.html)) has declined from 53% to 26%. The issue may not be young people’s [skills (or ostensive lack thereof)](https://theconversation.com/the-problem-isnt-unskilled-graduates-its-a-lack-of-full-time-job-opportunities-90104), but the lack of real opportunities for their skills to be utilised and developed further [in work contexts](https://theconversation.com/why-we-dont-need-to-prepare-young-people-for-the-future-of-work-98385).

What is new and innovative at the beginning of a five-year period is often commonplace, superseded or even obsolete by its end. Using digital technologies such as mobile phones, fit bits, social media, smart cards, rings, [glasses](https://www.wired.com/story/google-glass-2-is-here/) and watches, digital implants, QE codes, ‘the cloud’ and Artificial Intelligence (AI) assistants, we can monitor and manage many if not most aspects of our everyday and professional lives, as well as extend them though [augmented reality](http://www.realitytechnologies.com/augmented-reality) and [virtual reality](https://www.vrs.org.uk/virtual-reality/what-is-virtual-reality.html).

At other times, things which might have been anticipated to come to pass are still nascent, deferred or under further development, such as [automated marking of NAPLAN](http://cpl.asn.au/journal/semester-1-2018/robot-marking-automated-essay-scoring-and-naplan-a-summary-report) and other large cohort assessments more broadly, both with respect to technical aspects of their implementation and the broad polity.

Humans are both storytellers and a technological species; indeed, technology has been central to much of human evolution and progress, from pre-historic times, through history to modern times. With respect to archaeological evidence and historical developments of computational technology in mathematics, a detailed discussion paper ‘Mathematics, CAS technology and senior secondary mathematics education: issues and directions for implementation’ (September 2003) was presented in the committee papers to the predecessor of the VCAA’s Senior Secondary Curriculum and Assessment Committee (SSCAC) for the introduction of CAS technology into VCE Mathematics studies.

This paper outlined different forms of computational technology employed throughout human evolution and history, and how these had been developed in response to effectively solving types/classes of computational problems (broadly interpreted) in a range of practical contexts.

The 2003 discussion paper noted that the Greek root word *teckne* and the related Latin term *ars*, from which we get the English *technology* and *art* respectively, refer to both an *artefact* or device, and the means, or *art* of its *systematic* use. The thinking underpinning mathematics curriculum development has moved beyond a focus on specifying the technology artefact that is employed, to the range of mathematical functionality that can be deployed and used for various purposes, irrespective of the mode, device or platform by which this is implemented. Moreover, data, computational outputs, analysis and the like can be communicated and published using these technologies.

There are interesting examples and developments in the use of computational technology, from archaeological discoveries of regularities in marks on bones ([Ishango](https://en.wikipedia.org/wiki/Ishango_bone) and [Lebombo](https://en.wikipedia.org/wiki/Lebombo_bone)) from Paleolithic times, and through recorded history, such as ancient bamboo rod [computation tools for tax collection in China](https://www.nature.com/news/ancient-times-table-hidden-in-chinese-bamboo-strips-1.14482) (300BCE) to [bronze geared Greek navigational/astronomical tools](http://www.nature.com/news/2010/101124/full/468496a.html) (the [Antikythera mechanism](https://www.smithsonianmag.com/history/decoding-antikythera-mechanism-first-computer-180953979/), 200BCE), the devices of Pascal, Leibniz, Napier, Babbage, Bush and others, to, in the last five years, the hardware and software behind the [Rosetta satellite and Philae lander](https://www.space.com/27697-rosetta-comet-landing-full-coverage.html) voyage and landing on comet 67P/Churyumov-Gerasimenko, and the [Google Deep Minds Alpha Go](https://www.newscientist.com/article/2132086-deepminds-ai-beats-worlds-best-go-player-in-latest-face-off/) AI that in 2017 defeated the world Go champion in a three match series (and had previously beaten a range of grandmasters).

Chess (1997) had been dealt with much earlier by the [AI Deep Blue](https://www.wired.com/2017/05/what-deep-blue-tells-us-about-ai-in-2017/), but for the moment in information messy games such as poker humans seem to be [holding their own](http://www.sciencemag.org/news/2017/03/artificial-intelligence-goes-deep-beat-humans-poker) . AlphaGo won the ancient board game Go by learning from humans, but also by [machine learning](https://www.wired.com/2016/01/in-a-huge-breakthrough-googles-ai-beats-a-top-player-at-the-game-of-go/) involving neural networks. In the process it created [new strategies](https://deepmind.com/blog/innovations-alphago/) and knowledge that humans had not known or used previously. Go masters now have the delight of studying these innovations after millennia of established and well-known practice.

We live in a highly technological society whose vast range of digital technologies and AI interactions are part of our everyday lives, with applications (some established, some novel, others becoming commonplace), [personal digital assistants](https://www.predictiveanalyticstoday.com/top-intelligent-personal-assistants-automated-personal-assistants/) such as [Siri](https://www.apple.com/au/ios/siri/), [Cortana](https://www.microsoft.com/en-au/windows/cortana) and the like, clever [houses/smart homes](https://www.architecturelab.net/now-thats-smart-home-clever-house-tech-need/) (with functionality accessed and directed via mobile phone, as the pre-movie ads and announcements inform), car and driving assistants, autonomous vehicles (the mining industry, [La Trobe University](http://www.latrobe.edu.au/news/articles/2017/release/first-ever-trial-for-autonomous-shuttles)), AI in [healthcare](http://medicalfuturist.com/artificial-intelligence-will-redesign-healthcare/) and [legal services](http://www.abc.net.au/news/2017-11-13/artificial-intelligence-law-firm-without-lawyers-in-darwin/9146332), and now pay-to-mobile replacing tap and pay by card, with growing interest in [digital implants](https://www.theaustralian.com.au/business/technology/implant-technology-ive-got-you-and-lots-of-data-under-my-skin/news-story/ed400a21fa9250d326f74d037dd2e83b) for various functions, including hand-wave payment by [wearables](https://www.smh.com.au/technology/tap-and-go-smart-ring-makes-contactless-payments-fashionable-20180129-p4yyz3.html) or even [implants](https://www.smh.com.au/technology/tap-and-go-smart-ring-makes-contactless-payments-fashionable-20180129-p4yyz3.html).

Free online applications such as the computational engine [Wolfram Alpha](https://www.wolframalpha.com/) can be accessed from portable devices and can carry out sophisticated mathematical (such as those required for VCE Mathematics examinations) and other sorts of [computations](https://www.wolframalpha.com/tour/) across a broad range of fields, based on natural language queries, short programs and live data. These queries and their results can be shared via digital media including Facebook and Twitter.

All Victorian teachers and students have free access to Wolfram Alpha Pro ([via a DET licence](http://www.education.vic.gov.au/about/programs/learningdev/vicstem/Pages/wolframsoftware.aspx)), which provides step-by-step working for corresponding by hand computations for mathematical problems such as solving a system of simultaneous linear equations. Wolfram Alpha is powered by the computer algebra system Mathematica (also available through the DET licence) and can readily carry out computations for existing VCE Mathematics studies and examinations. In short, the sorts of single computations needed for VCE Mathematics can be done readily on a mobile phone.

In the media there is ongoing discussion of artificial intelligence, avatars, robotics, androids, cloning and [cybernetics](https://www.bbvaopenmind.com/en/article/the-future-of-artificial-intelligence-and-cybernetics/?fullscreen=true), algorithms, computational thinking, coding, power generation and home power, and the impact of these technologies on [work](http://www.abc.net.au/news/2017-08-09/artificial-intelligence-automation-jobs-of-the-future/8786962) ([could a robot do your job](http://www.abc.net.au/news/2017-08-08/could-a-robot-do-your-job-artificial-intelligence/)?), education and [lifestyle](http://www.abc.net.au/news/2017-08-07/artificial-intelligence-the-top-10-predictions-toby-walsh/8775034). From 1985 to 2015, there was a significant [shift in jobs and occupations](http://www.abc.net.au/news/2018-04-16/australia-on-track-1-million-new-jobs-since-2013-where-are-they/9597470). A recent McKinsey Global institute article [*Retraining and reskilling workers in the age of automation*](https://www.mckinsey.com/global-themes/future-of-organizations-and-work/retraining-and-reskilling-workers-in-the-age-of-automation) (January 2018) estimates that around 14 per cent of the global workforce may need to switch occupational categories due to the disruptive effects of digitisation, automation and advances in artificial intelligence. It also notes that companies will require different kinds of skills, with profound implications for individual career paths. The earlier McKinsey Global institute report ‘[A](https://www.mckinsey.com/global-themes/digital-disruption/harnessing-automation-for-a-future-that-works) [Future that Works: Automation, Employment and Productivity](https://www.mckinsey.com/~/media/McKinsey/Global%20Themes/Digital%20Disruption/Harnessing%20automation%20for%20a%20future%20that%20works/MGI-A-future-that-works_Full-report.ashx)’ (January 2017) notes that while few occupations are fully automatable, 60 per cent of all occupations have at least 30 per cent technically automatable activities.

The popular science fiction anthology television series [Black Mirror](https://en.wikipedia.org/wiki/Black_Mirror) (2011 to present) explores a series of alternative present or near future scenarios and is often challenging or confronting in relation to human–technology interaction, typically involving several of these elements. The show’s developer comments that the series of stand-alone episodes offer cautions about ‘the way we live now – and the way we might be living in 10 minutes' time if we're clumsy’*.* There have been the recent events in hacking and digital interference in elections, ‘fake news’ and the like through social media employing various call-bots and chat-bots, both internationally and locally. Likewise, there are concerns about the potential of [consumer privacy](https://www.welivesecurity.com/wp-content/uploads/2018/02/ESET_MWC2018_IoT_SmartHome.pdf?utm_source=wellenwide&utm_medium=post) for current smart devices to be activated remotely by third parties.

More broadly there is the notion of the [technological singularity](http://www.independent.co.uk/life-style/gadgets-and-tech/news/singularity-artificial-intelligence-humans-sexy-funny-ai-music-art-google-futurist-engineering-ray-a7633481.html), a conjectured event where growth in all of these and other developments leads to a stage where AI attain human intelligence level and beyond, and are autonomous and human independent. For some, such as [Ray Kurzweil](https://www.sciencealert.com/google-s-director-of-engineering-claims-that-the-singularity-will-happen-by-2029?perpetual=yes&limitstart=1), this would lead to a new and positive stage of human evolution, for others, including Elon Musk and the late Stephen Hawking, there are [concerns](https://www.businessinsider.com.au/autonomous-artificial-intelligence-is-the-real-threat-2015-9?r=US&IR=T) about humans being left behind in this context. There is no shortage of dystopian future scenarios developed from the *Matrix*, *Aliens* and *Terminator* movies, through to [*Elysium*](https://en.wikipedia.org/wiki/Elysium_%28film%29), *Ex Machina* (2015), [*Westworld*](https://en.wikipedia.org/wiki/Westworld_%28TV_series%29)(2016 – present), *Blade Runner* (1982, set in 2019), and [*Blade Runner 2049*](https://en.wikipedia.org/wiki/Blade_Runner) (2017, set in 2049). The median timeframe predicted by a range of futurists for the technology singularity is 2040–50. However, some futurists predict that technological singularity will occur as early as 2029, others believe it will take longer, while some are sceptical it will occur at all. There are a [range of views](http://www.businessinsider.com/predictions-for-after-singularity-2015-11?IR=T/#everything-is-going-to-change-1) as to what this might entail.

Philosophical and ethical considerations are deeply embedded in these discussions. Some local concerns have already arisen; recently, the Australian AI community requested that the Prime Minister advocate for a [ban on lethal autonomous weapons systems](http://www.smh.com.au/federal-politics/political-news/outlaw-killer-ai-robotics-experts-tell-malcolm-turnbull-20171105-gzfhqf.html) such as ‘[killer robots’](http://www.abc.net.au/news/2017-08-21/killer-robots-artificial-intelligence-tech-leaders-un-letter/8825906), an area of investigation in the military field. In 2017, Dubai introduced the first [robot police](https://www.telegraph.co.uk/news/2017/06/01/first-robotic-cop-joins-dubai-police/). On the other hand, there have been significant developments in [medical robotics](http://medicalfuturist.com/9-exciting-medical-robot-facts/), including [micro-robotics](https://phys.org/news/2016-07-remote-controlled-microrobots-medical.html).

What is clear is that through technology there is a merging of digital, physical and biological aspects of natural and human worlds. A challenge is how we work and potentially [co-evolve](http://platoandthenerd.org/blog/coevolution-of-human-and-artificial-intelligences) with these technologies. There is also a need for [critical, creative and ethical thinking](https://news.nd.edu/news/reilly-center-releases-its-2018-top-10-list-of-ethical-dilemmas-in-science-and-technology/) in relation to these developments, as discussed by Klaus Schwab, the founder and executive chair of the World Economic Forum in the [Fourth Industrial Revolution](https://www.weforum.org/agenda/2016/01/the-fourth-industrial-revolution-what-it-means-and-how-to-respond/). Mathematics is deeply involved in how we use and respond to these technologies and how we view, think about, and analyse them. In some regards the technical involvement of mathematics is increasingly more invisible, embedded in the sophistication of the design of the various algorithms and AI that are deployed in the technology. In other regards, the involvement of mathematics is becoming more visible as a means of rational inquiry to model and analyse such developments and their implications for society more generally.

3. The nature of the review work

In a fundamental sense the *practical* nature of this work is to look ahead for 5 years and 10 years, in a couple of steps, corresponding to two accreditation cycles. Central to the work of curriculum and assessment authorities are principled and coherent responses to the questions of: *What mathematics*? (Selection from culture, discipline and domain knowledge, theory and applications); *For whom*? (Subsets of the cohort, context); *How*? (Curriculum and assessment requirements and advice on possible pedagogies); and, *Why*? (Rationale and purpose).

To this we can add the question: *Who decides*? In Australia, states and territories have the constitutional responsibility for school education, set the curriculum and related assessment such as final year examinations and requirements, and monitor/supervise system specified school-based assessment. Such work takes place within the formal structures of the relevant authorities, boards and councils, informed by research, data and consultation.

In Victoria, students cannot leave school until they are 17 years old. *The Education and Training Reform Act 2006* includes:

* a requirement for all young people to participate in schooling (meaning in school or an approved equivalent) until they finish Year 10
* a requirement for all young people who have completed Year 10 to participate full-time (defined as at least 25 hours per week) in education, training or employment, or a combination of these activities, until they are 17 years old.

Senior secondary certificates sit in the broader context of the [National Partnership on Youth Attainment and Transitions](https://docs.education.gov.au/documents/national-partnership-youth-attainment-and-transitions-second-interim-evaluation-report). States and territories set targets for retention and participation in education, training or employment for this age group, such as 90 per cent in education, training or employment. The studies within a senior secondary certificate such as the VCE need to cater for a broad cohort of students, which form the majority of young people in the 17–19 year age group.

In Victoria, review or development of a senior secondary study or group of related studies for the VCE is undertaken by Expert Panels and [Review Panels](http://www.vcaa.vic.edu.au/Documents/vce/rolesresponsibilitiesofVCEreviewpanels09.pdf) convened and working in accordance with the [[VCAA’s Principles and guidelines for the development and review of VCE studies](https://www.vcaa.vic.edu.au/Documents/vce/2019_Principles_Procedures_VCE_review.docx)](https://www.vcaa.vic.edu.au/Documents/vce/2019_Principles_Procedures_VCE_review.docx), which were revised and updated in 2018, and general and specific terms of reference for that review. The Principles and Guidelines include the following VCE Curriculum Principles:

* optimise curriculum connections and pathways
* reflect democratic values and community standards
* balance challenge and expectation with the needs of the individual
* express and reflect enduring and dynamic aspects of a field or discipline

The VCE Mathematics review is undertaken in two stages. The first stage has been the formation and work of an Expert Panel to investigate possible structures for the VCE Mathematics group of studies, to identify in broad terms important related and to develop proposals for consultation. The second stage involves the formation of Review Panels to undertake the detailed development of the VCE Mathematics group of studies.

Both stages involve consultation with stakeholders and key interest groups. Given the significance of the depth of the review in terms of historical development of the study, comprehensive consultation on the proposed structure will be critical.

At its November 2017 meeting, SSCAC provided the following directions and terms of reference for the work of a VCE Mathematics Expert Panel for the first stage of the process. To develop a proposed structure for the VCE Mathematics group of studies, the Expert Panel will draw on selected papers and reports including:

* the ‘New Directions’ paper (September 2014)
* the ‘Senior Secondary Mathematics in Victoria: Working Towards Change’paper (October 2017)
* the contributed papers of experts nationally and internationally in response to the ‘Working Towards Change’ paper
* a meta-analysis of these papers
* benchmarking report of selected jurisdictions
* reports and papers such as ‘The Mathematical Sciences in 2025’.

Following consultation, SSCAC endorsement and Board approval of a particular structure and model, the Expert Panel will prepare a report for the Mathematics Review Panels outlining the structure of the study, the nature, purpose and scope of the component studies and the broad content to be developed for each of these studies.

Review Panels will then develop a draft VCE Mathematics proposal for each component study in accordance with the ‘VCAA Principles and Guidelines for the development and review of VCE studies’ and considering the report of the Expert Panel and the benchmarking reports, expert reports and selected papers. The role of the review panels is advisory and the proposed study designs will be subject to endorsement by SSCAC and approval by the Board of the VCAA.

In developing the component studies, Review Panels will consider:

* the relationship between the Victorian Curriculum and the study and the relationship between the study and post-schooling pathways
* the existence of overlap and/or duplication both within the study and between other VCE studies
* the relationship between the study and other VCE, VET and VCAL studies
* the extent to which the study reflects contemporary research and developments that are appropriate to the field of study and the senior secondary level of schooling
* the appropriateness and coherence of the structure of the study, and the areas of study and outcomes in relation to the aims of the study
* the capacity of the study to enable both broad participation and the achievement of excellence
* the effectiveness of the assessment program in measuring student performance against the learning outcomes
* the appropriateness of the weighting of the examination and school-based assessment
* the appropriateness of the areas of study and outcomes, and their relationship to the examination and school-based assessment
* the use of ICT and digital technologies in the study for the purposes of assessment
* the contribution of the study to the development of employability skills
* student workload
* gender and/or cultural and/or socioeconomic bias
* any specific minimum facilities and resources required to deliver the study.

The Review Panels will provide evidence that the following data have been considered in the review and evaluation of the study:

* enrolments and enrolment trends by unit and year level (including gender, sector and location)
* satisfactory completion rates by unit
* grade distributions
* reports from State Reviewer, Chairperson of the Examination Panel and Chief Assessor
* data from audit of school-based assessments
* benchmarking, expert and other reports and papers.

Consultation

In developing a proposal the Review Panels will ensure that, at the very least, they consult at the draft proposal stage with key interest groups and stakeholders including:

* Mathematics Association of Victoria (MAV)
* teachers and others on the VCE Study Register, interested in being consulted on VCE Mathematics
* the State Reviewers, Chairpersons of the Examination Panels and Chief Assessors
* the university and TAFE sectors
* peak representative agencies in Victoria
* teacher networks
* students.

The consultation draft and questionnaire will be made available on the VCAA website during the consultation period.

Curriculum is often described as having three key aspects:

* the *designed*, planned or intended curriculum, and related assessments, as specified by mandatory system documentation and processes, and supported by accompanying system provided advice and resources
* the *implemented* curriculum; what schools and teachers locally put in place as part of their teaching program to meet the requirements of the intended curriculum, including formal system school-based assessment requirements
* the *received* curriculum, what students experience in their learning in response to the planned and implemented curriculum, related formative and summative assessments at the school level, and formal external assessment, such as the [General Achievement Test](http://www.vcaa.vic.edu.au/Pages/vce/exams/gat/aboutgat.aspx) (GAT) and [examinations](http://www.vcaa.vic.edu.au/Pages/vce/exams/examsassessreports.aspx).

Pedagogical considerations are typically not part of the designed curriculum (which specifies the *what* but not the *how*); however, they play a key role in both implemented and received curriculums through ‘teaching and learning’. For this review, the term of reference for consultation explicitly includes seeking feedback from *students* as well as teachers and other stakeholders. That is, to directly seek current and recent-past student input, as well as proxy feedback through teachers. This could be done by explicitly involving current VCE students, as well as following up with recent former VCE students through the DET [On Track](http://www.education.vic.gov.au/about/research/Pages/ontrack.aspx) data, and surveys of post-secondary/undergraduate students. Indeed, as the Commonwealth Bank notes in its ‘[Jobs and Skills of the Future](https://rossdawson.com/wp-content/uploads/2017/11/Commonwealth-Bank_Jobs-and-Skills-of-the-Future-Report_November-2017.pdf)’ report (November 2017), *how* students learn will change as much as *what* students need to learn. This consideration is taken up in some depth in the recent [Gonski Report 2.0](https://docs.education.gov.au/system/files/doc/other/662684_tgta_accessible_final_0.pdf) ‘Through Growth to Achievement: Report of the Review to Achieve Educational Excellence in Australian Schools’ (March 2018).

The potential for technology in this process is discussed in some detail in ‘[Transforming Education](https://news.microsoft.com/uploads/prod/sites/66/2018/06/Transforming-Education-eBook_Final.pdf)’ (Microsoft, 2018).

While it is beyond the immediate scope of the review, careful consideration will need to be given to teacher background, professional learning and resources for practise that enable, and are congruent with, the aims and intentions of a revised senior secondary mathematics curriculum. This will entail advanced and complementary planning and appropriate resourcing to ensure teachers are suitably prepared to implement any significant developments in the curriculum that require new mathematical knowledge and skills and new [pedagogical content knowledge](https://www.narst.org/publications/research/pck.cfm).

In early 2018, the VCAA Mathematics Curriculum Manager received an email from a professional colleague of several decades; an experienced, well-regarded and capable secondary mathematics teacher and head of department, who retired at the end of that year:

I have been thinking about what the ideal school Maths Leader might look like in the 2020’s and 2030’s. Aside from having a high level of mathematical knowledge and a comprehensive understanding of the diverse body of the theory of mathematics education I wonder what further skills, knowledge and competencies will be required to execute the role of leader of a Mathematics department in a secondary school into the 2020’s and 2030’s.

It is an important issue as boomers retire in ever greater numbers and as Mathematics education changes so rapidly with the growth of technology. Further, what should a secondary Mathematics Department look like in the 2020’s and 2030’s? I would be interested to hear from you of your thoughts on this matter…

…the question results from a combination of musings and strategic planning for the future as I look towards retirement myself…

The contemporary conversations around quantum computing, travel to Mars, robotics, climate change, growing organs for replacing diseased or defective body parts and so on causes one to ask what mathematical competencies students will need in the next twenty years, what mind sets, what dispositions and training to negotiate the societies of the next 20 years. And what will mathematics teachers and maths leaders need to be able to do in the years ahead?

… my initial thinking includes what you have outlined but in addition the ideal leader will have a PhD in mathematics, will be able to lead staff into new territory confidently and fearlessly, be adaptable to new technologies, be intensely curious, flexible in attitude, understand the various modes of student comprehension in mathematics and so on.

The question I ask myself is this: what should my successor be like?

These deliberations go to the essence of why a deeper review of the mathematics curriculum is needed.

4. Mathematics

The question: ‘What is mathematics?’ is not an easy one to answer, given the long evolution of the discipline and its applications and the dynamic nature of its ongoing researches. Indeed, people have been developing, doing and using mathematics for much longer than consideration of it as a formal discipline, while historical mathematics has been around for millennia.

There are many quotes illustrative of [views and beliefs about mathematics](https://www.brainyquote.com/topics/mathematics) and its applications (see [TED maths talks](https://www.ted.com/topics/math)), often contradictory, espoused at various times over the past 200 years, for example:

‘The mathematics is not there till we put it there.’ Arthur Eddington.

‘God exists since mathematics is consistent, and the Devil exists since we cannot prove it.’ Andre Weil.

‘Pure mathematics is, in its way, the poetry of logical ideas.’ Albert Einstein.

‘The science of operations, as derived from mathematics more especially, is a science of itself, and has its own abstract truth and value.’ Ada Lovelace.

‘Mathematics is the most beautiful and most powerful creation of the human spirit.’ Stefan Banach.

‘God used beautiful mathematics in creating the world.’ Paul Dirac.

‘Mathematics is a game played according to certain simple rules with meaningless marks on paper.’ David Hilbert.

‘One of the most amazing things about mathematics is the people who do math aren't usually interested in application, because mathematics itself is truly a beautiful art form. It's structures and patterns, and that's what we love, and that's what we get off on.’ Danica McKellar.

‘Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality.’ Richard Courant.

‘We will always have STEM with us. Some things will drop out of the public eye and will go away, but there will always be science, engineering, and technology. And there will always, always be mathematics.’ Katherine Johnson.

‘It is not of the essence of mathematics to be conversant with the ideas of number and quantity.’ George Boole.

‘Logic and mathematics are nothing but specialised linguistic structures.’ Jean Piaget.

‘Many who have had an opportunity of knowing any more about mathematics confuse it with arithmetic and consider it an arid science. In reality, however, it is a science which requires a great amount of imagination.’ Sofia Kovalevskaya.

‘A mathematician is a blind man in a dark room looking for a black cat which isn’t there.’ Charles Darwin.

Math is really about the human mind, about how people can think effectively, and why curiosity is quite a good guide*.’* [William Thurston.](http://www.azquotes.com/author/31394-William_Thurston)

A particular maturation of the discipline followed from investigations into the philosophy of mathematics, including studies in the foundations of mathematics from the mid-nineteenth century through the first half of the twentieth century. These studies led to continuing investigations in the foundations and philosophy of mathematics; see, for example, *Philosophy of Mathematics* (Jaquette, 2002).

There are three well-known philosophies of mathematics:

* *Platonism* which holds there is an independent realm of mathematical objects that exist per se and that mathematical truth exists independently of time and is *discovered* by humans
* *Constructivism* which holds that mathematics is a very sophisticated and logically refined development of human language, where mathematical objects are human constructs, truth is part of historical time, established through rigorous human discourse, and mathematics is *invented*
* *Pragmatism*, which takes a different approach and considers mathematics in terms of what it does, a combination of *it is what it is* and *handsome is as handsome does.*

Each of these philosophies has its past and current adherents and influences the way that various social groups consider school mathematics and mathematics education, and what they argue for and about. As Paul Ernest notes: ‘…indeed it is argued throughout that the philosophy of mathematics, or philosophies of mathematics, underpin all mathematics curricula and teaching.’ (*Philosophy of Mathematics Education*, 1991, page 296).

It’s somewhat easier to identify various areas of mathematical activity, including established areas have had ongoing development and new that have areas arisen over the past 50 years. Irrespective of their philosophical inclinations, many mathematicians generally seem happy to take some things as given and get on with doing the business of mathematics.

In *Concepts of Modern Mathematics* (1975) mathematician Ian Stewart attempted to provide for the general reader an insight into modern approaches to mathematics and gives the following as key areas of mathematics: [analytic geometry](https://en.wikipedia.org/wiki/Analytic_geometry), [set theory](https://en.wikipedia.org/wiki/Set_theory), [abstract algebra](https://en.wikipedia.org/wiki/Abstract_algebra), [group theory](https://en.wikipedia.org/wiki/Group_theory), [topology](https://en.wikipedia.org/wiki/Topology) and [probability](https://en.wikipedia.org/wiki/Probability).

The journal [*Contemporary Mathematics – Fundamental Directions*](http://www.mathnet.ru/eng/cmfd) covers the following topics of contemporary mathematics: ordinary differential equations, partial differential equations, mathematical physics, real analysis and functional analysis, complex analysis, mathematical logic and foundations of mathematics, algebra, number theory, geometry, topology, algebraic geometry, Lie groups and the theory of representations, probability theory and mathematical statistics and discrete mathematics.

The Mathematics Subject Classification (MSC) covers two major mathematical reviewing databases, [Mathematical Reviews](https://en.wikipedia.org/wiki/Mathematical_Reviews) and [Zentralblatt MATH](https://en.wikipedia.org/wiki/Zentralblatt_MATH). The most recent version MSC2010 is currently being revised for release in 2020. It comprises five high-level organisers covering 97 areas:

[**General/foundations: study of foundations of mathematics and logic**](https://en.wikipedia.org/wiki/Mathematics_Subject_Classification#General/foundations_[Study_of_foundations_of_mathematics_and_logic])

General, including topics such as [recreational mathematics](https://en.wikipedia.org/wiki/Recreational_mathematics), [philosophy of mathematics](https://en.wikipedia.org/wiki/Philosophy_of_mathematics) and [mathematical modelling](https://en.wikipedia.org/wiki/Mathematical_model); [history](https://en.wikipedia.org/wiki/History_of_mathematics) and [biography](https://en.wikipedia.org/wiki/List_of_mathematicians); m[athematical logic](https://en.wikipedia.org/wiki/Mathematical_logic) and [foundations](https://en.wikipedia.org/wiki/Foundations_of_mathematics), including [model theory](https://en.wikipedia.org/wiki/Model_theory), [computability theory](https://en.wikipedia.org/wiki/Computability_theory), [set theory](https://en.wikipedia.org/wiki/Set_theory), [proof theory](https://en.wikipedia.org/wiki/Proof_theory), and [algebraic logic](https://en.wikipedia.org/wiki/Algebraic_logic).
[**Discrete mathematics/algebra: study of structure of mathematical abstractions**](https://en.wikipedia.org/wiki/Mathematics_Subject_Classification#Discrete_mathematics/algebra_[Study_of_structure_of_mathematical_abstractions])
C[ombinatorics](https://en.wikipedia.org/wiki/Combinatorics), o[rder theory](https://en.wikipedia.org/wiki/Order_theory), general [algebraic systems](https://en.wikipedia.org/wiki/Algebraic_system), [number theory](https://en.wikipedia.org/wiki/Number_theory), [field theory](https://en.wikipedia.org/wiki/Field_theory_%28mathematics%29) and [polynomials](https://en.wikipedia.org/wiki/Polynomial), [commutative rings](https://en.wikipedia.org/wiki/Commutative_ring) and [algebras](https://en.wikipedia.org/wiki/Commutative_algebra), [algebraic geometry](https://en.wikipedia.org/wiki/Algebraic_geometry), [linear](https://en.wikipedia.org/wiki/Linear_algebra) and [multilinear algebra](https://en.wikipedia.org/wiki/Multilinear_algebra)-[matrix theory](https://en.wikipedia.org/wiki/Matrix_%28mathematics%29), [associative rings](https://en.wikipedia.org/wiki/Associative_ring) and [associative algebras](https://en.wikipedia.org/wiki/Associative_algebra), [non-associative rings](https://en.wikipedia.org/wiki/Non-associative_ring) and [non-associative algebras](https://en.wikipedia.org/wiki/Non-associative_algebra), [category theory](https://en.wikipedia.org/wiki/Category_theory), [homological algebra](https://en.wikipedia.org/wiki/Homological_algebra), [K-theory](https://en.wikipedia.org/wiki/K-theory), [group theory](https://en.wikipedia.org/wiki/Group_theory) and generalisations, [topological groups](https://en.wikipedia.org/wiki/Topological_group), [Lie groups](https://en.wikipedia.org/wiki/Lie_group), and analysis upon them.
[**Analysis: study of change and quantity**](https://en.wikipedia.org/wiki/Mathematics_Subject_Classification#Analysis_[Study_of_change_and_quantity])

R[eal functions](https://en.wikipedia.org/wiki/Real_function), including [derivatives](https://en.wikipedia.org/wiki/Derivative) and [integrals](https://en.wikipedia.org/wiki/Integral), m[easure](https://en.wikipedia.org/wiki/Measure_%28mathematics%29) and [integration](https://en.wikipedia.org/wiki/Integral), [complex functions](https://en.wikipedia.org/wiki/Complex_function), including [approximation theory](https://en.wikipedia.org/wiki/Approximation_theory) in the [complex domain](https://en.wikipedia.org/wiki/Complex_number), [potential theory](https://en.wikipedia.org/wiki/Potential_theory), [several complex variables](https://en.wikipedia.org/wiki/Several_complex_variables) and [analytic spaces](https://en.wikipedia.org/wiki/Analytic_space), [special functions](https://en.wikipedia.org/wiki/Special_functions), [ordinary differential equations](https://en.wikipedia.org/wiki/Ordinary_differential_equation), [partial differential equations](https://en.wikipedia.org/wiki/Partial_differential_equation), [dynamical systems](https://en.wikipedia.org/wiki/Dynamical_system) and [ergodic theory](https://en.wikipedia.org/wiki/Ergodic_theory), [difference equations](https://en.wikipedia.org/wiki/Difference_equation) and [functional equations](https://en.wikipedia.org/wiki/Functional_equation), [sequences](https://en.wikipedia.org/wiki/Sequence), [series](https://en.wikipedia.org/wiki/Series_%28mathematics%29), [summability](https://en.wikipedia.org/wiki/Summability), [approximations](https://en.wikipedia.org/wiki/Approximation_theory) and [expansions](https://en.wikipedia.org/w/index.php?title=Expansion_(approximation_theory)&action=edit&redlink=1), [harmonic analysis](https://en.wikipedia.org/wiki/Harmonic_analysis), including [Fourier analysis](https://en.wikipedia.org/wiki/Fourier_analysis), [Fourier transforms](https://en.wikipedia.org/wiki/Fourier_transform), [trigonometric approximation](https://en.wikipedia.org/wiki/Trigonometric_approximation), [trigonometric interpolation](https://en.wikipedia.org/wiki/Trigonometric_interpolation), and [orthogonal functions](https://en.wikipedia.org/wiki/Orthogonal_function), abstract [harmonic analysis](https://en.wikipedia.org/wiki/Harmonic_analysis), [integral transforms](https://en.wikipedia.org/wiki/Integral_transform), [operational calculus](https://en.wikipedia.org/wiki/Operational_calculus), [integral equations](https://en.wikipedia.org/wiki/Integral_equation), [functional analysis](https://en.wikipedia.org/wiki/Functional_analysis), including [infinite-dimensional holomorphy](https://en.wikipedia.org/wiki/Infinite-dimensional_holomorphy), [integral transforms](https://en.wikipedia.org/wiki/Integral_transform) in [distribution spaces](https://en.wikipedia.org/wiki/Distribution_space), [operator theory](https://en.wikipedia.org/wiki/Operator_theory), c[alculus of variations](https://en.wikipedia.org/wiki/Calculus_of_variations) and [optimal control](https://en.wikipedia.org/wiki/Optimal_control); [optimisation](https://en.wikipedia.org/wiki/Optimization_%28mathematics%29) (including [geometric integration theory](https://en.wikipedia.org/wiki/Geometric_integration_theory)).

[**Geometry and topology: study of space**](https://en.wikipedia.org/wiki/Mathematics_Subject_Classification#Geometry_and_topology_[Study_of_space])

[Geometry](https://en.wikipedia.org/wiki/Geometry), [convex geometry](https://en.wikipedia.org/wiki/Convex_geometry) and [discrete geometry](https://en.wikipedia.org/wiki/Discrete_geometry), [differential geometry](https://en.wikipedia.org/wiki/Differential_geometry), [general topology](https://en.wikipedia.org/wiki/General_topology), [algebraic topology](https://en.wikipedia.org/wiki/Algebraic_topology), [manifolds](https://en.wikipedia.org/wiki/Manifold), [global analysis](https://en.wikipedia.org/wiki/Global_analysis), [analysis on manifolds](https://en.wikipedia.org/wiki/Analysis_on_manifolds) (including [infinite-dimensional holomorphy](https://en.wikipedia.org/wiki/Infinite-dimensional_holomorphy)).

[**Applied mathematics /other: study of applications of mathematical abstractions**](https://en.wikipedia.org/wiki/Mathematics_Subject_Classification#Applied_mathematics_/_other_[Study_of_applications_of_mathematical_abstractions])
[Probability theory](https://en.wikipedia.org/wiki/Probability_theory), [stochastic processes](https://en.wikipedia.org/wiki/Stochastic_processes), [statistics](https://en.wikipedia.org/wiki/Statistics), [numerical analysis](https://en.wikipedia.org/wiki/Numerical_analysis), [computer science](https://en.wikipedia.org/wiki/Computer_science), [mechanics](https://en.wikipedia.org/wiki/Mechanics) (including [particle mechanics](https://en.wikipedia.org/wiki/Particle_mechanics)), [mechanics of deformable solids](https://en.wikipedia.org/w/index.php?title=Mechanics_of_deformable_solids&action=edit&redlink=1), [fluid mechanics](https://en.wikipedia.org/wiki/Fluid_mechanics), [optics](https://en.wikipedia.org/wiki/Optics), [electromagnetic theory](https://en.wikipedia.org/wiki/Electromagnetic_theory), classical [thermodynamics](https://en.wikipedia.org/wiki/Thermodynamics), [heat transfer](https://en.wikipedia.org/wiki/Heat_transfer), [quantum theory](https://en.wikipedia.org/wiki/Quantum_mechanics), [statistical mechanics](https://en.wikipedia.org/wiki/Statistical_mechanics), structure of matter, [relativity](https://en.wikipedia.org/wiki/Theory_of_relativity) and [gravitational theory](https://en.wikipedia.org/wiki/Gravitational_theory), including [relativistic mechanics](https://en.wikipedia.org/wiki/Relativistic_mechanics), [astronomy](https://en.wikipedia.org/wiki/Astronomy) and [astrophysics](https://en.wikipedia.org/wiki/Astrophysics), [geophysics](https://en.wikipedia.org/wiki/Geophysics), [operations research](https://en.wikipedia.org/wiki/Operations_research), [mathematical programming](https://en.wikipedia.org/wiki/Mathematical_programming), [game theory](https://en.wikipedia.org/wiki/Game_theory), [economics](https://en.wikipedia.org/wiki/Mathematical_economics), [social](https://en.wikipedia.org/wiki/Mathematical_sociology) and [behavioural sciences](https://en.wikipedia.org/wiki/Mathematical_psychology), [biology](https://en.wikipedia.org/wiki/Biology) and other [natural sciences](https://en.wikipedia.org/wiki/Natural_science), [systems theory](https://en.wikipedia.org/wiki/Systems_theory); control, including [optimal control](https://en.wikipedia.org/wiki/Optimal_control), [Information](https://en.wikipedia.org/wiki/Information) and [communication](https://en.wikipedia.org/wiki/Communication), [circuits](https://en.wikipedia.org/wiki/Electrical_network), [mathematics education](https://en.wikipedia.org/wiki/Mathematics_education).

In its 2016 report, ‘[The Mathematical Sciences in Australia A Vision for 2025](https://www.science.org.au/files/userfiles/support/reports-and-plans/2016/mathematics-decade-plan-2016-vision-for-2025.pdf)’, the Australian Academy of Sciences gives the following broad characterisation:

*What are the mathematical sciences? The term ‘mathematical sciences’ is used to encompass mathematics, statistics and the range of mathematics-based disciplines including teaching, teacher education and educational research.*

Two fundamental and complementary aspects of mathematics are its abstraction and its applicability, and the interrelated notions of proof and computation. In the paper ‘[Solutions for a Complex Age Long Range Plan for Mathematical and Statistical Sciences Research in Canada 2013–2018](http://longrangeplan.ca/wp-content/uploads/2012/12/3107_MATH_LRP-1212-web.pdf)’the Natural Sciences and Engineering Research Council (NSERC) identifies (page 5) a range of contexts for applications of mathematics from different fields within the discipline (the mathematics behind every-day and not-so-every-day things):

**Algebra and Number Theory**: CDs and DVDs, cryptography and secure communications (cell phones and banking), Google’s Page Rank® system, 3D modelling and animation software;

**Geometry and Topology**: robotics, GPS systems, sensor networks, space mission design;

**Real and Complex Analysis**: aircraft design, wireless, radio and TV (electromagnetic spectrum), MRIs, spectroscopy;

**Partial** **Differential Equations**: waves, systems modelling (wound healing, tumour growth and tsunami prediction), radar imaging, MRIs and CT scans, finance;
**Combinatorics**: quantum theory, Internet server connections, scheduling, analysis of networks;

**Numerical Analysis**: MRIs, imaging and compressed sensing, signal processing;
**Functional Analysis and Operator Algebras**: quantum information theory, modelling and design, machine learning;

**Spectral Theory**: spectroscopy; and

**Optimization and Control**: electromechanical systems (cars, aircraft, power grids), logistics, feasibility studies.

Similarly, for Probability and Statistics (page 6):

**Probability and Stochastics:** biology, medicine, finance, stock markets, economics, epidemic spread, fluid flow;

**Biostatistics:** drug development, population genetics, nutrition, environmental health, population health, assessment of medical treatments, clinical trials;

**Networks and Graphs:** psychology, education, sociology, causality;

**Extreme Value Theory:** finance, climate, hydrology, insurance;

**Spatial and Space-Time Models and Methods:** weather prediction, data compression, ergonomics, cartography, geology, machine learning;

**Functional Data:** fluid dynamics, growth curves, climate change, biomechanics, geophysics;
**Data Mining and Machine Learning:** astronomy, pharmacokinetics, bioinformatics, robotics, artificial intelligence, high-energy physics;

**Surveys and Sampling:** animal abundance studies, consumer behaviour, political opinion polls, population health; and

**Actuarial Science:** pensions, health policy, risk, insurance and demography.

The US National Academies Press 2013 paper ‘[The Mathematical Sciences in 2025](http://www.eu-maths-in.eu/EUMATHSIN/wp-content/uploads/2016/02/2013-USA_report_mathematics_2025.pdf)’ notes (page 2) that:

Mathematical sciences work is becoming an increasingly integral and essential component of a growing array of areas of investigation in biology, medicine, social sciences, business, advanced design, climate, finance, advanced materials, and many more. This work involves the integration of mathematics, statistics, and computation in the broadest sense and the interplay of these areas with areas of potential application. All of these activities are crucial to economic growth, national competitiveness, and national security, and this fact should inform both the nature and scale of funding for the mathematical sciences as a whole. Education in the mathematical sciences should also reflect this new stature of the field.

It also (page 4) characterises that the mathematical sciences:

 … aim to understand the world by performing formal symbolic reasoning and computation on abstract structures. One aspect of the mathematical sciences involves unearthing and understanding deep relationships among these abstract structures. Another aspect involves capturing certain features of the world by abstract structures through the process of modelling, performing formal reasoning on the abstract structures or using them as a framework for computation, and then reconnecting back to make predictions about the world. Often, this is an iterative process. Yet another aspect is to use abstract reasoning and structures to make inferences about the world from data. This is linked to the quest to find ways to turn empirical observations into a means to classify, order, and understand reality—the basic promise of science.

Through the mathematical sciences, researchers can construct a body of knowledge whose interrelations are understood and where whatever understanding one needs can be found and used. The mathematical sciences also serve as a natural conduit through which concepts, tools, and best practices can migrate from field to field.

This latter point is also noted in the 2010 UK report ‘[International Review of Mathematical Sciences](https://www.epsrc.ac.uk/newsevents/pubs/international-review-of-mathematical-sciences/)’ which observes (page 10):

The mathematical sciences form an essential part of science, engineering, medicine, industry and technology and serve as one of the pillars of education at all levels. Major contributions to the health and prosperity of society arise from insights, results and algorithms created by the entire sweep of the mathematical sciences, ranging across the purest of the pure, theory inspired by applications, hands-on applications, statistics of every form and the blend of theory and practice embodied in operational research.

The report also highlights the importance of recognising aspects of *unity of work* in the mathematical sciences (page 10):

A long-standing practice has been to divide the mathematical sciences into categories that are, by implication, close to disjoint. Two of the most common distinctions are drawn between ‘pure’ and ‘applied’ mathematics, and between ‘mathematics’ and ‘statistics’. These and other categories can be useful to convey real differences in style, culture and methodology, but in the Panel’s view, they have produced an increasingly negative effect when the mathematical sciences are considered in the overall context of science and engineering, by stressing divisions rather than unifying principles. Furthermore, such distinctions can create unnecessary barriers and tensions within the mathematical sciences community by absorbing energy that might be expended more productively. In fact, there are increasing overlaps and beneficial interactions between different areas of the mathematical sciences.

As well as the importance of abstraction and transfer from one field to another (page 11):

Much of the content of mathematical sciences research is expressed in terms of general relationships such as equations, formulae and diagrams, which constitute a universal language for expressing abstractions in science, engineering, industry and medicine. Without the convenient forms provided by the mathematical sciences, other disciplines might develop their own specialised terminology for mathematical concepts, thereby inhibiting communication outside their field. Happily, the same (or closely related) abstractions can provide insights and intellectual tools for understanding and solving problems that seem disparate when represented in their ‘natural’ form. For example, the mathematical ideas in modelling the flow around swimming multicellular organisms might seem far removed from designing the Internet or controlling congested traffic because experts in each area use a different vocabulary; but underlying similarities, as well as differences, are often revealed through mathematical abstractions, which also serve to translate fundamental concepts with different names from one field to another.

Mathematics not only comprises its ‘stuff’ – the recorded questions, discourse, written and digital literature, conjectures, results and theorems – but also a body of active practice involving individuals and communities. In addition to the traditional questions in the philosophy of mathematics: What is the nature and basis of mathematical knowledge? What is the existence, form and nature of mathematical objects? Paul Ernest in the *Philosophy of Mathematics Education* (Ernest, 1991, page 25) adds other questions, including:

* what is the purpose of mathematics?
* what is the role of human beings in mathematics?
* how does the subjective knowledge of individuals become the objective knowledge of mathematics?
* how has mathematical knowledge evolved?
* how does its history illuminate the philosophy of mathematics?
* what is the relationship between mathematics and other areas of human knowledge and experience? And
* why have the theories of pure mathematics proved to be so powerful and useful in their applications to science and to practical problems?

Indeed, Ernest sees an adequate account of the genesis of mathematical knowledge as an essential element of any philosophy of mathematics. Thus, he articulates the following criteria for an adequate philosophy of mathematics. It should account for:

* mathematical knowledge: its nature, justification and genesis
* the objects of mathematics, their nature and origins
* the applications of mathematics: its effectiveness in science, technology and other realms
* mathematical practice: the activity of mathematicians, both in the present and the past.

Nowadays through the media and internet there is much sharing of views, opinions, general discussion and research about mathematics and its practices, such as the blogs of [Keith Devlin](https://profkeithdevlin.org/) and [Terry Tao](https://terrytao.wordpress.com/), and others, including discussion on [gender in mathematics](https://www.theatlantic.com/science/archive/2016/11/math-women/506417/).

Experimental mathematics

Central to Paul Ernest’s analysis are two key questions: What is mathematical truth? and on what basis do we warrant this? Alternatively, one could ask: How do we obtain and establish mathematical knowledge? This statement has been carefully phrased to be neutral with respect to two key philosophical positions: is mathematical knowledge intrinsically there, part of the hardware of the universe and hence to be *discovered* (the Platonist view) or is it *constructed*, part of the software of humanity, and hence to be invented (the constructivist view).

The [experimental tradition in mathematics](https://www.maa.org/external_archive/devlin/devlin_03_09.html) dates from ancient times and has [continued throughout](https://en.wikipedia.org/wiki/Experimental_mathematics) human history. During the latter half of the nineteenth century and the first half of the twentieth century investigations into the foundations of mathematics and the trend toward formal axiomatisation of mathematics, stimulated by developments in mathematical logic, set theory and computability theory, were predominant. This trend had led to a conceptualisation of mathematics that emphasised its hypothetical-deductive nature and the role of proof, rather than the experimental approach, or at least general acknowledgment of the latter’s contribution.

As John Crossley notes in his paper ‘Axioms’ (Monash University, 1976, page 11) with respect to the axiomatisation of number systems:

The historical order was essentially the reverse of the logical order required to build up the complex number system … this last is a point which many of us – and this goes for me as a teacher as much as for anybody – tend to neglect: the fact that when one starts doing mathematics seriously one proceeds in the reverse to the logical order. One tries to analyse notions to see what is going on … then one eventually gets to suitable axioms … the axioms come last in the actual development of mathematics. One tries to explain something in terms of simpler and simpler notions and eventually (after a certain amount of work) one may come out with axioms. It is only at this stage that one reverse the whole process and provides logical definitions of the familiar properties and new properties.

How students are typically *taught* in school is fundamentally based on the logical order of mathematics (given *this* and *this* then *that* and *that*), whereas how they *learn* mathematics often has much more to do with their direct experience and experiment.

In foundations of mathematics both the notions of truth (semantic) and proof (syntactic) are recognised and the relationship between them is a field of study in itself: mathematical logic and meta-mathematics.

Historically, what have been identified as mathematical truths have arisen through a combination of experiment, intuition and proof, yet little of the notion and practise of *experiment* is typically incorporated into the implemented curriculum. The advent of sophisticated mathematical technologies since the 1970s has given both renewed interest and vigour to the experimental approach, as noted by Keith Devlin in his 2009 paper ‘[What is Experimental Mathematics?](https://www.maa.org/external_archive/devlin/devlin_03_09.html)’:

Experimental mathematics is the name generally given to the use of a computer to run computations – sometimes no more than trial-and-error tests – to look for patterns, to identify particular numbers and sequences, to gather evidence in support of specific mathematical assertions, that may themselves arise by computational means, including search.

Had the ancient Greeks (and the other early civilizations who started the mathematics bandwagon) had access to computers, it is likely that the word "experimental" in the phrase "experimental mathematics" would be superfluous; the kinds of activities or processes that make a particular mathematical activity "experimental" would be viewed simply as mathematics. On what basis do I make this assertion? Just this: if you remove from my above description the requirement that a computer be used, what would be left accurately describes what most, if not all, professional mathematicians have always spent much of their time doing!

Many readers, who studied mathematics at high school or university but did not go on to be professional mathematicians, will find that last remark surprising. For that is not the (carefully crafted) image of mathematics they were presented with. But take a look at the private notebooks of practically any of the mathematical greats and you will find page after page of trial-and-error experimentation (symbolic or numeric), exploratory calculations, guesses formulated, hypotheses examined, etc.

The reason this view of mathematics is not common is that you have to look at the private, unpublished (during their career) work of the greats in order to find this stuff (by the bucketful). What you will discover in their published work are precise statements of true facts, established by logical proofs, based upon axioms (which may be, but more often are not, stated in the work).

Because mathematics is almost universally regarded, and commonly portrayed, as the search for pure, eternal (mathematical) truth, it is easy to understand how the published work of the greats could come to be regarded as constitutive of what mathematics actually is. But to make such an identification is to overlook that key phrase "the search for". Mathematics is not, and never has been, merely the end product of the search; the process of discovery is, and always has been, an integral part of the subject. As the great German mathematician Carl Friedrich Gauss wrote to his colleague Janos Bolyai in 1808, ‘It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment.’

In fact, Gauss was very clearly an "experimental mathematician" of the first order. For example, his analysis – while still a child – of the density of prime numbers, led him to formulate what is now known as the Prime Number Theorem, a result not proved conclusively until 1896, more than 100 years after the young genius made his experimental discovery.

For most of the history of mathematics, the confusion of the activity of mathematics with its final product was understandable: after all, both activities were done by the same individual, using what to an outside observer were essentially the same activities – staring at a sheet of paper, thinking hard, and scribbling on that paper. But as soon as mathematicians started using computers to carry out the exploratory work, the distinction became obvious, especially when the mathematician simply hit the ENTER key to initiate the experimental work, and then went out to eat while the computer did its thing. In some cases, the output that awaited the mathematician on his or her return was a new "result" that no one had hitherto suspected and might have no inkling how to prove.

What makes modern experimental mathematics different (as an enterprise) from the classical conception and practice of mathematics is that the experimental process is regarded not as a precursor to a proof, to be relegated to private notebooks and perhaps studied for historical purposes only after a proof has been obtained. Rather, experimentation is viewed as a significant part of mathematics in its own right, to be published, considered by others, and (of particular importance) contributing to our overall mathematical knowledge. In particular, this gives an epistemological status to assertions that, while supported by a considerable body of experimental results, have not yet been formally proved, and in some cases may never be proved. (It may also happen that an experimental process itself yields a formal proof. For example, if a computation determines that a certain parameter *p*, known to be an integer, lies between 2.5 and 3.784, that amounts to a rigorous proof that *p* = 3.)

When experimental methods (using computers) began to creep into mathematical practice in the 1970s, some mathematicians cried foul, saying that such processes should not be viewed as genuine mathematics – that the one true goal should be formal proof. Oddly enough, such a reaction would not have occurred a century or more earlier, when the likes of Fermat, Gauss, Euler, and Riemann spent many hours of their lives carrying out (mental) calculations in order to ascertain "possible truths" (many but not all of which they subsequently went on to prove). The ascendancy of the notion of proof as the sole goal of mathematics came about in the late nineteenth and early twentieth centuries, when attempts to understand the infinitesimal calculus led to a realization that the intuitive concepts of such basic concepts as function, continuity, and differentiability were highly problematic, in some cases leading to seeming contradictions. Faced with the uncomfortable reality that their intuitions could be inadequate or just plain misleading, mathematicians began to insist that value judgments were hitherto to be banished to off-duty chat in the university mathematics common room and nothing would be accepted as legitimate until it had been formally proved.

What swung the pendulum back toward (openly) including experimental methods was in part pragmatic and part philosophical. (Note that word "including". The inclusion of experimental processes in no way eliminates proofs.)

The pragmatic factor behind the acknowledgment of experimental techniques was the growth in the sheer power of computers, to search for patterns and to amass vast amounts of information in support of a hypothesis.

At the same time that the increasing availability of ever cheaper, faster, and more powerful computers proved irresistible for some mathematicians, there was a significant, though gradual, shift in the way mathematicians viewed their discipline. The Platonist philosophy that abstract mathematical objects have a definite existence in some realm outside of Mankind, with the task of the mathematician being to uncover or discover eternal, immutable truths about those objects, gave way to an acceptance that the subject is the product of Mankind, the result of a particular kind of human thinking.

The shift from Platonism to viewing mathematics as just another kind of human thinking brought the discipline much closer to the natural sciences, where the object is not to establish "truth" in some absolute sense, but to analyse, to formulate hypotheses, and to obtain evidence that either supports or negates a particular hypothesis.

Stephen Wolfram has argued similarly from the perspective of computational thinking in his book [*A New Kind of Science*](http://www.wolframscience.com/nks/) (2002) and has developed a [blog](http://blog.stephenwolfram.com/) on various mathematics related and other topics, including an interactive cloud-based article ‘[Two hours of experimental mathematics](http://blog.stephenwolfram.com/2017/03/two-hours-of-experimental-mathematics/)’. From the Australian context, Jonathan Borwein (1951–2016) Laureate Professor of Mathematics at University of Newcastle, in collaboration with David Bailey, have been international leaders and [key contributors](https://experimentalmath.info/blog/) in the field of experimental mathematics for over two decades.

A key consideration for curriculum design is how the experimental approach could be effectively and suitable incorporated (or not) in curriculum, pedagogy and assessment.

5. Mathematics education

Mathematics education and the philosophy of mathematics education are younger fields, see, for example *T*h*e Philosophy of Mathematics Education* (Ernest, 1991) with compulsory universal secular education beginning in the latter part of the nineteenth century in what are now regarded as liberal democratic developed nations.

A key question in the philosophy of mathematics education is: why do we include mathematics as part of the school curriculum? Why is it compulsory (or not) at various stages of primary and secondary school education? Paul Ernest addresses this question in one way in his paper ‘[Why Teach Mathematics](https://www.scribd.com/document/46697739/Why-Teach-Mathematics-Paul-Ernest)?’ From educational systems the response to this is typically framed in statement(s) relating to a rationale (why) and a set of aims (what), as can be found in the introduction to the VCE Mathematics study design, and in comparable documents from other systems or jurisdictions. To this one can also add the question, *who* gets to decide and on what *authority*? In most jurisdictions this is an Authority, Board, Council, Department or Ministry, or some combination or variation of these, according to the relevant constitutional arrangements. In Victoria it is the Victorian Curriculum and Assessment Authority (VCAA) empowered by the [*Education and Training Reform Act (2006)*](http://www.education.vic.gov.au/about/department/legislation/Pages/act2006.aspx), Section 2.5.

The [Rationale and Aims for Victorian Curriculum Mathematics](http://victoriancurriculum.vcaa.vic.edu.au/mathematics/introduction/rationale-and-aims) Foundation–Level 10 provide a concise articulation of aims for school mathematics education

The Mathematics curriculum aims to ensure that students:

* develop useful mathematical and numeracy skills for everyday life, work and as active and critical citizens in a technological world
* see connections and apply mathematical concepts, skills and processes to pose and solve problems in mathematics and in other disciplines and contexts
* acquire specialist knowledge and skills in mathematics that provide for further study in the discipline
* appreciate mathematics as a discipline – its history, ideas, problems and applications, aesthetics and philosophy.

School mathematics

School mathematics is not the same as mathematics more generally, as indicated in the introduction to the ‘[Statements of Learning for Mathematics’(2006)](http://www.curriculum.edu.au/verve/_resources/SOL_Mathematics_2006.pdf):

Mathematics educators acknowledge that there are different perspectives on the philosophy of mathematics and mathematics education and that these underpin the relationship between mathematics, the domain of school mathematics and the mathematics curriculum of any school system. While States and Territories have expressed a range of views on mathematics and mathematics education, with corresponding diverse interpretation and representation in curriculum design, a review of principal curriculum documents reveals significant alignment across the following broad underlying themes.

**Mathematics is dynamic**
Mathematical knowledge has developed across cultures throughout history and continues to develop today. Mathematics education responds to social change, developments in mathematics, new technologies and new approaches to mathematical inquiry.

**Mathematics is an integral part of a general education**Mathematics is part of our cultural heritage. All students have a right to learn mathematics and the language of mathematics, to make sense of mathematics, to be confident in their use of mathematics and to see how it can help them make sense of their world and the worlds of others. High expectations for achievement, conceptual understanding and the opportunity to learn reasonable and challenging mathematics are fundamental to equity and social justice.

**Mathematics contributes to individual and collective development**
Mathematics and the capacity to be numerate, that is, the ability to effectively apply mathematics in everyday, recreational, work and civic life, is vital to the quality of participation in society.

**Mathematics connects with other curriculum areas**Mathematics is a domain that supports learning and application in other curriculum areas and also draws on them for learning contexts.

The ‘Statements of Learning for Mathematics’ and their professional elaborations draw upon the following aims, which are a synthesis of those from mathematics curriculums across Australia at that time, and intended to provide students with the opportunity to develop:

Knowledge and understanding of concepts and ideas, and facility with mathematical skills and processes across key areas of mathematics with:

* mental and written computation and numerical reasoning
* function and pattern, generalisation, logical and algebraic reasoning
* the identification and measurement of attributes or characteristics of shapes, objects, data and chance events
* geometric reasoning and the visualisation, representation, location and transformation of shapes and objects in space

The capacity and disposition to deploy mathematical knowledge, understanding, skills and processes in a range of situations through:

* using and building on prior knowledge, generalising to other contexts, making conjectures and incorporating new information into existing structures
* posing and solving problems, mathematical modelling, developing proofs and conducting investigations − thinking creatively, generating alternatives when solving problems, and working individually and cooperatively
* reflecting upon and discussing mathematical ideas, problems and processes, to formulate and test their own solutions, and have these tested by others
* evaluating representations of mathematical information and challenging mathematical ideas by considering purpose and point of view

The capacity to communicate effectively through:

* the use of informal and formal mathematical language to convey, logically and clearly, their mathematical understandings, thinking and reasoning in oral, electronic and written media
* representation of their mathematical ideas and reasoning in different ways which reflect their conceptual understandings for various audiences and purposes
* the selection and effective use of a range of mathematical strategies, models, information and communication technologies and related critical literacies

Enjoyment of mathematics and confidence in the use of mathematics in everyday situations through appreciation of:

* its relevance as part of their personal and working lives
* its nature as a dynamic, diverse and complex domain with interwoven and interconnected concepts
* the nature of mathematical thinking and its historical and cultural roles.

These are a reasonable set of aspirations, and the preceding discussion shows how justification for the inclusion of mathematics as a study in the compulsory curriculum is couched in social, cultural, pragmatic, utilitarian and aesthetic dimensions, and encompasses reciprocity, complementarity and balance in the relation between benefit to the individual and society. In practice the extent to which these aspirations are even partially achieved is variable, and they are often related to economic notions of productivity and competitiveness in a global context.

In education in Victoria, the study of mathematics in the senior secondary years has always been a matter for *student choice* (although in the original 1990 VCE, students had to select a study from the Mathematics, Science and Technology group of studies), albeit with strong encouragement in respect of the importance and/or desirability of continuing with some mathematical studies in this stage of schooling, scaffolded by tertiary level pre-requisite requirements. Indeed, by enrolment VCE Mathematics is the largest group of VCE studies.

Thus, the rationale and aims for the VCE Mathematics study do not need to offer the same degree of argument for inclusion, rather they articulate a view of the discipline itself, and its role in a general education, as described in the scope, rational and aims in the introduction to the VCE Mathematics study design (2016):

**Scope of study**
Mathematics is the study of function and pattern in number, logic, space and structure, and of randomness, chance, variability and uncertainty in data and events. It is both a framework for thinking and a means of symbolic communication that is powerful, logical, concise and precise. Mathematics also provides a means by which people can understand and manage human and natural aspects of the world and inter-relationships between these. Essential mathematical activities include: conjecturing, hypothesising and problem posing; estimating, calculating and computing; abstracting, proving, refuting and inferring; applying, investigating, modelling and problem solving.

**Rationale**
This study is designed to provide access to worthwhile and challenging mathematical learning in a way which takes into account the interests, needs, dispositions and aspirations of a wide range of students, and introduces them to key aspects of the discipline. It is also designed to promote students’ awareness of the importance of mathematics in everyday life in a technological society, and to develop confidence and the disposition to make effective use of mathematical concepts, processes and skills in practical and theoretical contexts.

**Aims**This study enables students to:

* develop mathematical concepts, knowledge and skills
* apply mathematics to analyse, investigate and model a variety of contexts and solve practical and theoretical problems in situations that range from well-defined and familiar to open-ended and unfamiliar
* use technology effectively as a tool for working mathematically.

Mathematics is presented as a discipline with its own nature, discourse and ways of working. These notions and considerations are a natural part of the discourse of various social groups and their beliefs, values and aims, and a central aspect of how feedback from consultation is expressed, and for what purposes. Beliefs, values and aims have commonalities but also vary between individuals and social groups.

Paul Ernest draws on the psychological work of Perry (1970) on epistemological and ethical positions, combines it with the work of Williams (1961) and Cosin (1972) on social groups and aims of education with respect to connecting social groups with ideologies and various aspects of mathematics education. In this context an ideology is a coherent and comprehensive set of constructs that draws together an identifiable combination of values, beliefs and ideas. For Ernest, the critical aspect of this is that ideologies are *competing* belief systems, and he notes that they are often seen as ‘the way things really are’ by their adherents, because they are a medium in which relations of power subsist. Another way of saying this is that belief systems give rise to the ‘natural attitude’ for those of the social group(s) that support them, that is, belief systems can tend to be confounded with ‘truth’. These in turn underpin their responses to situations where genuine alternatives or choices of action are possible.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **Social Group** | **Industrial trainer** | **Technological pragmatist** | **Old humanist** | **Progressive educator** | **Public educator** |
| Political ideology | Radical right or new right | Meritocratic conservative | Conservative/Liberal | Liberal | Democratic socialist |
| View of mathematics | Set of truths and rules | Unquestioned body of useful knowledge | Body of structured pure knowledge | Process view: personalised mathematics | Social constructivism |
| Moral values | Authoritarian ‘Victorian’ values: choice, effort, self-help, work, moral weakness, us-good, them-bad | Utilitarian, pragmatism, expediency, wealth creation,technological development | Blind justice, objectivity, rule-centred structure,hierarchy,paternalistic classical view | Person-centred,Caring, empathy, human values, nurturing, maternalistic, romantic view | Social justice, liberty, equality, fraternity, social awareness, engagement and citizenship |
| Theory of society | Rigid hierarchy, market-place | Meritocratic hierarchy | Elitist, class stratified | Soft hierarchy, welfare state | Inequitable hierarchy, needing reform |
| Theory of the child | Elementary school tradition: child ‘fallen angel’ and ‘empty vessel’ | Child ‘empty vessel’ and ‘blunt tool’Future worker or manager | Dilute elementary school viewCharacter buildingCulture tames | Child-centred,Progressive view,Child: ‘growing flower’ and ‘innocent savage’ | Social conditions view: ‘clay moulded by environment’ and ‘sleeping giant’ |
| Theory of ability | Fixed and inheritedRealised by effort | Inherited by ability | Inherited cast of mind | Varies, but needs cherishing | Cultural product: not fixed |
| Mathematical aims | ‘Back to basics: numeracy and social training in obedience | Useful maths to appropriate level and certification (industry centred) | Transmit body of mathematical knowledge (maths centred) | Creativity, self-realisation through mathematics (child centred) | Critical awareness and democratic citizenship via mathematics |
| Theory of learning | Hard work, effort, practice, rote | Skill acquisition, practical experience | Understanding and application | Activity, play, exploration | Questioning, decision making, negotiation |
| Theory of teaching mathematics | AuthoritarianTransmission, drill, ‘no frills’ | Skill instructorMotivate through work relevance | Explain, motivatePass on structure  | Facilitate personal explorationPrevent failure | Discussion, conflict, questioning of content and pedagogy |
| Theory of resources | Chalk and talk only, anti-calculator | Hands-on and micro-computers | Visual aids to motivate | Rich environment to explore | Socially relevantauthentic |
| Theory of assessment in mathematics | External testing of simple basics | Avoid cheatingExternal tests and certificationSkill profiling | External examinations based on hierarchy | Teacher led internal assessmentAvoid failure | Various modesUse of social issues and content |
| Theory of social diversity | Differentiated schooling by class, crypto-racist, mono-culturalist | Vary curriculum by future occupations | Vary curriculum by ability only (maths neutral) | Humanise neutral maths for all: use local culture | Accommodation of social and cultural diversity a necessity |

*The Philosophy of Mathematics Education* (Ernest, 1991, page 137)

As Ernest notes, any such classification is open to critique, social groups and ideologies overlap and evolve, and vary in their currency and influence. They are mutable rather than fixed.

In practice, multiple ideologies influence aspects of curriculum design, development, evaluation and review, and are more (or less) influential at various stages of schooling and related transitions, and in different socio-political circumstances. These considerations provide awareness of how various social groups, of themselves and in combination, may contribute to discussion of mathematics curriculum, for example, while digital technologies are routinely and widely used in business, research, industry and academia, for doing and applying mathematics, there are a range of views related to their use in mathematics education.

The following is an extension of the previous table that shows how views of the various social groups could be related to the role of digital technology to school mathematics education:

|  |  |  |
| --- | --- | --- |
| **Social group** | **Key aim for mathematics education** | **Beliefs, values and choices (preferences)** |
| **View of the use of technology in mathematics and mathematics education** | **Positive use of technology** | **Concerns for use of technology** |
| Industrial trainer | To ensure that the student population has a minimum level of mathematical competency in the domain of mathematical literacy (numeracy) to be able to compete effectively in the labour market and contribute to productivity gains | Cautious and/or neutral | Affirms the use of technology such as arithmetic or scientific calculators, spreadsheets and the like where these contribute to the effective development of technology independent numeracy skills, and enhance the capacity to conduct work related transactions and computations reliably and accurately | May lead to dependency on calculator technology for basic calculations for a significant proportion of the student population, and hence reduce the capacity for independent facility to carry out such computations. |
| **Old humanist** | The development of a subset of the student population who are suitably prepared for further mathematical studies as the potential next generation of mathematicians | Reserved and/or negative | Can be a useful tool to support high level pure mathematical research, given a sound technology free basis for the development of underpinning mathematical knowledge and skills | Has significant potential to challenge the view of mathematics as an essentially human mental activity, may lead to a diminution of capability with certain valued mental and by hand skills |
| **Technological pragmatist** | The development of a flexible and broader range of problem solving strategies, both with and without the use of technology | Pragmatic and/or positive | Enhances capacity to model situations and solve a broader range of problems in application contexts | The effective use of technology depends on the development of related meta-cognitive skills, such as formulation, anticipation of the reasonableness of results, interpretation of results. Technology may be used indiscriminately without this supporting structure |
| **Progressive educator** | To ensure that the individual is able to respond to the challenges and benefits of technology | Selective and/or positive | Technology can be used to scaffold student learning and enable some students to access mathematical knowledge and skills that would otherwise likely not have been readily accessible | Excessive use of technology may result in the dehumanisation of social and cultural discourse. Students diminish valued social interactive with technology- based activities |
| **Public educator** | To ensure that the individual is able to contribute critically and constructively to social discourse  | Integrated and/or positive | Technology is a fundamental aspect of social dynamics, and the integrated use of technology enables students to be empowered as problem solvers and models the nature of society | Differential access to technology can lead to problems of social justice such as access and equity both in terms of educational opportunity  |

*The Philosophy of Mathematics Education* (Ernest, 1991, page 137)

The utility of mathematics is often presented as a key justification for its study; however, students often ask teachers: *why* do I need to study this? *when* and *where* will I use it? and so on. As society changes, so do the notions of utility, nevertheless it is possible to consider three broad levels of utility:

* mathematics that is used necessarily and naturally by most people on an everyday basis, almost everyone needs to know and do it
* mathematics that is used regularly by various groups of people or occupations, but not everyone needs to know and do it
* mathematics that is highly specialised, only a small number of people are required to have high-level expertise and ability to do it.

The first two levels relate to the first two aims of the Victorian Curriculum Mathematics:

* develop useful mathematical and numeracy skills for everyday life, work and as active and critical citizens in a technological world
* see connections and apply mathematical concepts, skills and processes to pose and solve problems in mathematics and in other disciplines and contexts

The utility of mathematics is generally supported in the broad sense; close to 95 per cent of VCE students enrol in a Units 1 and 2 level mathematics study and over 85 per cent enrol in a Units 3 and 4 level mathematics study.

Not surprisingly, there are always critiques and questions about how effectively this utility is aligned to the real needs and interest of students and society more generally, and how efficaciously it is developed in relation to relevant and real contexts. In terms of deciding what content is included in the curriculum, and the various offerings within the senior secondary curriculum, these three levels provide useful criteria for assisting decision-making about what to include (or not), for whom and for what purposes.

Other key reasons teachers give for the study of mathematics are that it:

* enables further mathematical study (that is, the purpose for learning *this* mathematics is so that one can subsequently learn *that* mathematics)
* teaches reasoning
* has an aesthetic dimension – for example, beauty, elegance and the like.

These relate to the second two aims of the Victorian Curriculum Mathematics:

* acquire specialist knowledge and skills in mathematics that provide for further study in the discipline
* appreciate mathematics as a discipline – its history, ideas, problems and applications, aesthetics and philosophy.

The ‘enabling further mathematical study’ argument is generally acknowledged, although there is reasonable critique about ensuring that it is applied in a necessary and sufficient way and is not overdone, and that there is some flexibility in how students might acquire relevant background.

For some, it is almost axiomatic that the study of mathematics ‘teaches reasoning’, but this argument is also open to some critical scrutiny. It needs to be robustly interrogated, especially as a general default when viewed from the perspective of the received curriculum across the whole of the student cohort. The benefits that may well have accrued from such study for established and successful teachers of mathematics, and *some* of their students, are not necessarily those experienced by most students. There is also the question as to what ‘mathematical reasoning’ really encompasses, ideally and in practice, and how deliberately and effectively this is part of teaching and learning experience at school.

The important of repeated practise of algorithms for various mathematical computations to achieve suitable level of proficiency and fluency, which is a large part of the received curriculum, entails a level of recognition and procedural understanding, as well as developing some robustness. That is, students need to identify that a certain procedure is applicable and to implement it efficiently and effectively.

Students also become aware through their work with variables and other mathematical objects, that general case arguments are an important aspect of mathematical reasoning. That is, based on commonalities (objects, definitions, conditions, constraints and the like), and on seeing various questions as particular cases of a mathematical system or structure, for which general results have been established, certain classes of problems can be identified and dealt with accordingly.

The hypothetical-deductive nature of this aspect of mathematical reasoning (accepting these givens and using certain logical principles, then this and that follows) can be illustrated by well-known examples in different topics of the curriculum involving number, algebra, functions, probability, geometry and the like. Ultimately, these examples are well-rehearsed knowns. The ability to formulate ‘new’ conjectures that may or may not be true and develop possible proofs for this, is something that students have typically had less exposure to, and experimental approaches have a role to play in investigating possibilities for such conjectures.

Thought needs to be given to the creative aspects of proof, and the nature of [mathematical discourse](https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/) around this, if these approaches are to be more fully developed in the senior secondary mathematics curriculum. Proof is seen as one of the key activities of mathematics; however, mathematicians make mistakes from time to time, and like to optimise effective use of their time, so even in this area there are interesting connections with technology such as [model checkers](https://people.mpi-sws.org/~rupak/Papers/SoftwareModelChecking.pdf), [automated proof checking](https://dspace.mit.edu/handle/1721.1/85424), [proof assistants](https://en.wikipedia.org/wiki/Proof_assistant) (human-machine interactive applications) and [automated theorem proving](https://link.springer.com/article/10.1007/s10817-016-9377-1). These developments can be seen to have some heritage from the notion of a philosophical language and related ideas, through the work of Leibniz, Boole, Frege, Russell and Whitehead, Hilbert to Church, Godel, Turing and others.

Modelling and problem-solving are well popularised forms of mathematical reasoning that have been incorporated into the senior secondary mathematics curriculum for several decades, with some success, although there is always room for improvement. In most instances ‘modelling and problem-solving’ involves the application of known solutions and approaches in contexts that are somewhat open-ended and unfamiliar.

The aesthetic justification is an interesting one, often with appeal to those who are established in the discipline and its history, as some of the earlier quotes on mathematics indicate. Indeed, at the elite level in any human enterprise or activity, be it sport, arts or mathematics, there are things which are held to have some aesthetic capital, although this often based on agreed acclamation ‘in the context’. For mathematics, descriptions such as [‘elegance’, ‘austere beauty’](https://en.wikipedia.org/wiki/Mathematical_beauty), and the like are variously used. Apart from the matter of beauty being in the eye of the beholder, this is perhaps not an aspect which would be prominent in the recollections of many students; however, it is a proposition that feedback can be gathered on. The insight and intuition which underpins new applications of mathematics speaks to an inclination and capacity to connect things in previously unthought of ways, often in response to new stimuli or challenges.

Some of these stimuli and challenges arise in circumstances involving uncertainty and randomness in a wide range of fields such as [biology](https://en.wikipedia.org/wiki/Biology), [chemistry](https://en.wikipedia.org/wiki/Chemistry), [ecology](https://en.wikipedia.org/wiki/Ecology), [neuroscience](https://en.wikipedia.org/wiki/Neuroscience), [physics](https://en.wikipedia.org/wiki/Physics), as well as [image processing](https://en.wikipedia.org/wiki/Image_processing), [signal processing](https://en.wikipedia.org/wiki/Signal_processing), [information theory](https://en.wikipedia.org/wiki/Information_theory), [computer science](https://en.wikipedia.org/wiki/Computer_science), [cryptography](https://en.wikipedia.org/wiki/Cryptography) and [telecommunications](https://en.wikipedia.org/wiki/Telecommunications). Thinking about such matters is informed by data and reasoning in probability and statistics (stochastic reasoning) that has its own distinctive nature and style.

In the preceding discussion and commentary, mathematics has among other things been referred to as a science, an art, a way of viewing the world, a way of reasoning and a special type of language. In part mathematics reflects aspects of each of these and has its own distinctive characteristics.

STEM and critical and creative thinking

**S**cience **T**echnology **E**ngineering and **M**athematics ([STEM](https://en.wikipedia.org/wiki/Science%2C_technology%2C_engineering%2C_and_mathematics)) has been a key driver in policy and practice in educational systems nationally and internationally over the past decade, especially the last five years, for example the current Victorian Government’s STEM Initiative [VicSTEM](http://www.education.vic.gov.au/about/programs/learningdev/vicstem/Pages/default.aspx). The former and current Chief Scientists, [Ian Chubb](http://www.chiefscientist.gov.au/wp-content/uploads/STEM_AustraliasFuture_Sept2014_Web.pdf) and [Alan Finkel](http://www.chiefscientist.gov.au/2016/03/report-australias-stem-workforce/), and others have been strong in STEM advocacy such as the [National STEM School Education Strategy (2015)](https://www.education.gov.au/national-stem-school-education-strategy-2016-2026). Some participants in the discourse have attempted to broaden STEM to include, for example, Medicine ([STEMM](https://www.swinburne.edu.au/media/swinburneeduau/about-swinburne/docs/pdfs/Defining-STEMM-at-Swinburne.pdf)), the Arts ([STEAM](http://stemtosteam.org/)), and other variations, or interpret the components more broadly (to include a range of mathematics, science and technology areas, for example environmental science, general mathematics or food technology). A dominant perspective, however, has been one which includes only particular ‘hard’ or ‘advanced’ studies such as Mathematical Methods, Specialist Mathematics, Physics, Chemistry, Computer Science and the like (Engineering per se is not a VCE study, although engineering is part of some VET courses). This conceptualisation of STEM would not be out of place in the traditional two maths (pure and applied), two sciences (physics and chemistry) and English combination undertaken by ‘maths–science’ stream students in Year 12 in the 1970s. On the other hand, a broader or ‘soft’ interpretation of STEM would likely include other studies with substantive aspects of science, technology and mathematics such as Biology, Environmental Science, Further Mathematics and [Informatics](https://www.ed.ac.uk/files/atoms/files/what20is20informatics.pdf). Others have concerns about a focus on advanced STEM having a negative impact on other disciplines and areas of the curriculum. Sir Kenneth Robinson is well-known for arguing strongly for the critical role of [the arts in education](https://www.theage.com.au/national/victoria/learning-salsa-as-important-as-maths-education-expert-says-20180327-p4z6f8.html) , while others look to combine STEM with the Arts as [STE(A)M](https://research.acer.edu.au/cgi/viewcontent.cgi?article=1299&context=research_conference).

The traditional or ‘advanced’ STEM perspective has long had a dominant role in senior secondary mathematics curriculum and assessment and continues to be highly influential.

There is much attention in the media with respect to system performance on international tests such as TIMSS and PISA, and trends in enrolments in the advanced mathematics and science courses. A robust supply of students and graduates with backgrounds in these studies/disciplines is clearly vital for a modern technological society in developed countries; however, there is genuine debate about what the circumstances really are, and what this means. Some have argued that there is a [STEM crisis](https://www.livescience.com/43296-what-is-stem-education.html), or at least a potential crisis evolving, with students not choosing advanced STEM subjects in sufficient enabling numbers in senior secondary schooling or undergraduate study for the future needs of society and the economy.

Others argue that [there is no crisis](https://spectrum.ieee.org/at-work/education/the-stem-crisis-is-a-myth), indeed there is a [sufficient supply in general](https://news.rutgers.edu/qa/there-stem-worker-shortage-rutgers-professor-debates-issue-national-academies/20140310#.WdLSvrpuI2x), with various areas of need arising in the market from time to time. Indeed, where expertise is required in certain areas, it is argued that the natural market signals such as salary level are appropriate and effective, and a range of reports indicate that there has been no marked increase in STEM areas salaries that would typically indicate under-supply in the market.

Some care needs to be taken with these discussions and arguments, especially as enrolments in senior secondary certificate studies (where students have choice) are a zero-sum game – increased enrolments in one field/area correspond to decreased enrolments in other areas, yet multiple priorities are often applied with some competition or tension between them. Students are highly sensitive to mark signals from entrance scores such as the ATAR score as well as market signals such as historical and likely employment opportunity, income, and esteem/standing or popular image of occupation. For example, in terms of student perceptions for achieving a high ATAR score, the study of some languages may be seen as likely to provide a better ‘return’ than undertaking study of, say, Specialist Mathematics.

There are both general and specific arguments for the role of STEM more broadly. Some aspects of learning from STEM fields/disciplines are important and valuable enablers of rational inquiry for an informed critical citizenry, for example the scientific method and stochastic reasoning.

On the other hand, approaches that guide or direct students to certain areas run the real risk of making, or at least being suggestive of, promises that may not be realisable. The October 2017 [Graduate Outcome Survey – Longitudinal (GOS-L)](https://www.qilt.edu.au/docs/default-source/gos-reports/2017-gos-l/2017-gos-l-national-reportbb518791b1e86477b58fff00006709da.pdf?sfvrsn=2bb9e33c_2) by the Social Research Centre at the Australian National University includes data on short and medium term full-time employment outcomes by study area.

This is quite distinct from enabling and informing opportunity, such as the work of the Australian Mathematical Sciences Institute [CHOOSEMATHS](http://choosemaths.org.au/) program. One of the key issues for STEM is [gender representation](http://www.sciencegenderequity.org.au/about/what-is-sage/) in STEM fields, at school, undergraduate and graduate levels and in the workforce. For current VCE Units 3 and 4 mathematics studies, there is an even ratio of female to male enrolments in Further Mathematics, a 45:55 ratio in Mathematical Methods, and a 38:62 ratio in Specialist Mathematics.

School maths leaders and teachers

As indicated in the earlier email from the now retired mathematics head of faculty, it is important to consider enduring and dynamic aspects of a field such as mathematics, traditional and contemporary examples, contexts and settings, problems and issues, and the future orientation of the field.

The distribution of the population of teachers in general and mathematics teachers is one with a distribution skewed to older teachers, as data from the *Staff in Australia’s Schools 2013 Survey* (Table 3.2, page 15) shows:



For secondary mathematics, in 2013 the proportions of teachers that were aged less than 35, 36–50 and 51 or greater was 24: 36: 40, with an average age of 46 years. Five years on, the consequent category transitions will have taken place, with some new graduates entering the field and older teachers retiring. With respect to new entrants to teaching there are concerns about employment and attrition rates, as discussed in the AITSL report: [What do we know about early career teacher attrition rates in Australia?](https://www.aitsl.edu.au/docs/default-source/research-evidence/spotlight/spotlight---attrition.pdf?sfvrsn=40d1ed3c_0) (August 2016).

A teacher who had commenced their secondary mathematics teaching career in 1980 would now be around 60 years of age with several years until retirement. They would have learnt their school and undergraduate mathematics in an era where by hand computation was the most efficient approach for a range of problems, and the availability and functionality of digital technologies was much more limited. Education in general and mathematics education in particular have strong conservative traditions, with a natural focus on what is, or is perceived to be (or desired to be) enduring, more so than the discipline itself which is highly dynamic. The nineteenth and early twentieth centuries led to views of mathematics as an abstract discipline in its own right (so called pure mathematics) and these views had a significant impact on the construction of mathematics school curriculums in the 1960s and 1970s, an era when many of those who are now latter career stage mathematics leaders learnt their mathematics, formed their initial views on mathematics and their beliefs on what is to be valued in mathematics education.

This was an era before the development of, and widespread access to, powerful digital technologies for mathematical computation, and prior to the emergence in the 1990’s of social constructivism as an alternative philosophy of mathematics to Platonism and pragmatism.

Rather than believing that mathematical objects and truth exist independently of humans, constructivism sees mathematics as being actively created by human insight, intuition, logic and refinement, much like any other field or discipline. From a Platonist perspective, the effectiveness of mathematics in comprehending the universe is because it is hard-wired (in some special but perhaps unknowable way) into the fabric of the universe, the thought stream of the mind of the creator of the universe as it were. For the constructivist, the effectiveness of mathematics is it is a way of looking at things humans have constructed together to comprehend the universe and their place in it, and to clearly see and understand the connected human enterprises.

Technology has been a significant stimulus to the discipline, both in its demands for mathematics to enable developments in a technological world, but also in the impact of computational technologies in mathematics itself. As with other disciplines, digital technologies as communications tools have enabled international collaboration on mathematical investigations, modelling and problem-solving. While the technology based experimental approach to mathematics had yet to emerge, there were some efforts in the primary and early/mid secondary mathematics curriculum during the mid to late 1980s through the work of [Seymour Papert](https://tltl.stanford.edu/content/seymour-papert-s-legacy-thinking-about-learning-and-learning-about-thinking), with Logo, Turtle and Mindstorms.

Notwithstanding the opportunities and challenges presented by widespread access to such technologies (and changing views on their acceptability for various purposes within the school curriculum), they are used widely outside school wherever mathematics is applied, and for the discourse surrounding such research and applications.

6. Meta-analysis of views, ideas and approaches

In Section 3 it was noted that central to the work of curriculum and assessment authorities are principled and coherent responses to the questions of: *What mathematics*? (Selection from culture, discipline and domain knowledge, theory and applications); *For whom*? (Subsets of the cohort, context); *How*? (Curriculum and assessment requirements and advice on possible pedagogies); and *Why*? (Rationale and purpose).

Many societies and cultures have contributed and continue to contribute to the growth of mathematics, often in times of scientific, technological, economic, artistic and philosophical change and development. Complementary to this broad perspective of mathematics are the various mathematical practices that take place every day in communities around the world. mathematics can be described in terms of its objects, what they are and how they came to be; its established body of knowledge and why this is held to be true; its effective application in science, technology and other domains; and the practice and activities of mathematicians’ past and present. Mathematical knowledge includes knowledge of concepts, objects, definitions and structures. A small collection of mathematical ideas, objects, and structures, and the relationships between these, is taken as [undefined](http://pandora.nla.gov.au/pan/129125/20121206-0015/vels.vcaa.vic.edu.au/maths/glossary.html#undefined) and given in a context.

New mathematical objects, structures and relationships are developed from these, moving from simple to more complex and sophisticated ideas and practices. The motivation for accepting certain things as given building blocks for mathematical knowledge may be initially related to intuitive understanding of ideas and objects experienced with respect to the natural or human worlds. These and their subsequent developments are not [empirical](http://pandora.nla.gov.au/pan/129125/20121206-0015/vels.vcaa.vic.edu.au/maths/glossary.html#empirical) knowledge but abstract mathematical entities.

The abstract nature of mathematics gives rise to its applicability in a wide range of contexts, as mathematical objects, structures and relationships do not depend on context for their existence but are interpreted to model key features of these contexts.

Mathematical reasoning and thinking includes problem-posing, problem-solving, investigation and modelling. It encompasses the development of algorithms for computation, formulation of problems, making and testing conjectures, and the development of abstractions for further investigation. Computation and proof are essential and complementary aspects of mathematics that enable students to develop thinking skills directed toward explaining, understanding and using mathematical concepts, structures and objects. They provide a framework for the development of mathematical skills and techniques exemplified in the use of algorithms for computation and for the development of general case arguments.

Congruence between curriculum (planned, delivered, received), pedagogy (teaching and learning) and assessment (formative and summative) is vital for effective implementation that aims to achieve any set of goals. These are inter-dependent aspects of the educational endeavour.

For curriculum developers, this involves clear articulation of what students are expected to know and be able to do, based on various historical, analytical and research evidence in mathematics, mathematics education and developments in society more generally, as well as policy settings and directions. What are the key constructs, structures and relations? What are the givens, and based on what and whose choices, values and preferences?

In the senior secondary context, where the selection of study pathways (apart from the [English requirements](http://www.vcaa.vic.edu.au/Pages/wtn/vce/about.aspx)) is a matter of student choice, this means considering the nature, purpose and scope of possible offerings in relation to the student cohort and its sub-groups. For teachers, it involves content knowledge from the discipline, knowledge of applications, pedagogical knowledge and pedagogical content knowledge, as well as ‘knowing their students’. It also requires awareness and familiarity with suitable resources and ongoing professional learning. For students, it involves their natural curiosity, engagement and inclination to learn, dispositions with respect to their learning, including playfulness, inquiry, diligence and resilience, as well as their responsiveness to various intrinsic and extrinsic influences and motivations, pragmatic, aspirational or otherwise.

Student agency

This is presented first for consideration, as in the past it has often not been so prominent. The notion of student agency applies across and within the designed, implemented and received curriculum. Curriculum statements of aims, goals, intentions, aspirations and the like often refer to entitlements (couched in terms of prospective agency as critical and informed citizens in a modern democratic liberal society) and what one would like to see as outcomes for students; however, the mathematics education research literature indicates that for many students these are not necessarily well realised. Rather student experiences are often reported to be those of anxiety, disenfranchisement, disappointment and disengagement. Indeed, many students likely view their mathematical studies as something that is probably good for you, but not necessarily enjoyable. The substantial continued rise in enrolments in Further Mathematics (roughly two out of three VCE students at Year 12) speaks to students voting with their enrolments to endorse a study where real-life applicability is close at hand, and where they have ready access to the relevant enabling technology without restriction for computation.

As several invited respondents observe, students should be actively engaged in an ongoing narrative about the what, why and wherefore of their mathematical studies. Nowadays students are much less passive recipients of received wisdom than earlier generations. While senior secondary students have by and large not developed an informed overview of the scope, complexity, demand, inter-relationship and applications of mathematics that could potentially be studied, they do have their interests and see mathematical connections.

There is a sense of student agency, where their views and feedback have some input into future directions at the design level, such as during this current review. Traditionally, student agency has occurred indirectly by proxy where teachers put forward summary views on behalf of their students (*my* students think that…). Student voices can sometimes be accessed indirectly and in the first person through commentary and discussions on student blogs such as [ATAR Notes Online Student Community.](https://atarnotes.com/) Commentary from current students and feedback from past students of several years ago, indicating how their mathematical studies actually enabled progress in their subsequent studies and/or work, provides for some student agency with respect to the designed curriculum.

As remarked earlier selection of studies for a VCE pathway is a matter for student choice. This sense of agency is akin to seeing a movie: one can choose from the list of available options, then one watches the movie, for better or worse, unless one decides to leave early. Agency *within* a study is referred to by the first student respondent (a VCE Year 12 student who completed all three sequences of mathematics) as well as the second student respondent (employed for several years post VCE and graduation, considering topicality/relevance of contexts for mathematical study in relation to future study/work). The selection of a sequence of one or more mathematics studies/units will often depend on other factors such as university prerequisites, for example Mathematical Methods Units 3 and 4. For a completely prescribed ‘high-stakes’ study such as Mathematical Methods Units 3 and 4, once chosen there is limited opportunity for student agency, possibly in relation to some minor elements of school-based assessment, depending on task design. For other studies such as Foundation Mathematics Units 1 and 2, although completely prescribed, the structure of this study is explicit about the use of approaches and contexts that actively engage student agency for the application of mathematics, and the assessment of student learning. For studies such as General Mathematics Units 1 and 2, Specialist Mathematics Units 1 and 2 and Further Mathematics Units 3 and 4, which have a core/required and module/selected topic structure, there is opportunity for engagement of student agency. However, in many cases this choice of topics/modules seems to be exercised by the school and/or the classroom teacher.

Student agency is also evident in the implemented and received curriculums, and in both cases is, in some respects, co-related with teaching approaches. Other respondents emphasised the desirability of independent student thinking; creativity and curiosity, posing problems/problem elicitation; and exploration and investigation. This agency can really only develop in contexts where there is opportunity for its exercise. This can be achieved partly through pedagogical approaches, but also through the expectations and opportunities of the school-based component of the formal assessment system, through tasks with some unfamiliar and open-ended aspects where students make decisions and exercise some choice about context, data, directions, conditions, approaches and techniques. In terms of the received curriculum, many of today’s student actively and widely use a range of online resources beyond the traditional teacher and text combination, including blogs, forums, [video sites](https://www.youtube.com/khanacademy), [Eddie Woo](https://www.youtube.com/channel/UCq0EGvLTyy-LLT1oUSO_0FQ) (who videos his maths lessons), [open courses](https://ocw.mit.edu/courses/mathematics/), and third party resources such as [ITUTE](http://www.itute.com/download-free-vce-maths-resources/) and the like.

There is tension between student agency within a study and external expectations for the quantum of delivered content in terms of affordances for post-secondary study. For example, the purpose of calculus-based studies is seen to be preparatory for further study in mathematics or disciplines that depend substantially on such mathematics, and coverage of preparatory content is a priority.

Some curriculum structures

Within Victoria (see Paper 2: ‘Senior Secondary Mathematics in Victoria: Working Towards Change’) and other jurisdictions around the world, there have generally been between two and six different levels of mathematics ‘studies’, subjects’ or ‘courses’ for the senior secondary years. To some extent this depends on the liberal, general or vocational focus of the relevant certification, and the intended sub-set of the student cohort.

These are typically tiered, informally or formally, with respect to standing for tertiary entrance purposes such as qualifying for contribution towards a tertiary entrance score or ranking such as the [ATAR](http://www.vtac.edu.au/results-offers/atar-explained.html). This has often been the case where systems have sought to distinguish between vocational, general and academic mathematics subjects, for example Group 1 and Group 2 subjects pre-VCE. Since the inception of the VCE, all studies count towards the ATAR.

Courses are often differentiated by the level and extent of pure mathematical challenge provided by the content and corresponding computational demands (with and without technology). In some jurisdictions this is accompanied by provisions that selection of one study/subject/course precludes that of another, they are mutually exclusive. In each case there is a corresponding rationale for the suite of offerings and their combinations (or not). This has often been associated with the extent to which abstract mathematical formalism is part of the study/subject/course, including functions, algebra and calculus, and the extent to which practical contexts for application are embedded, and the nature and range of these applications. There are also ‘this leads to that’ considerations in terms of ‘suitable preparation’, ‘assumed knowledge and skills’ or ‘prerequisites’ and how these enable or prevent mathematical study pathways involving various subject sequences, combinations or proscriptions.

Victorian and Australian structure and models

In Australian senior secondary certificates have generally had three levels of senior secondary mathematics subjects that count towards an ATAR score, and in some jurisdictions either a variation on the least demanding of these, or an alternative subject or subjects that do not count for ATAR, with a range of different statuses, depending on the jurisdiction. While there has always been good broad alignment across the different states and territories, the AC work 2010–2012 and its incorporation into subsequent state and territory senior secondary review cycles has enhanced this further, so there are essentially four types of subjects (each with a four-unit sequence) at increasing levels of (pure) mathematical demand as one goes down the list:

Essential Mathematics

General Mathematics

Mathematical Methods

Specialist Mathematics (Mathematical Methods is a pre- or co-requisite subject).

The nature of the application contexts also varies, from everyday real life to business and industry to STEM(M), economics and other areas that need technical mathematics support, to advanced pure and applied mathematics. Some jurisdictions proscribe certain combinations such as General Mathematics + Mathematical Methods (NSW) while others allow them (Victoria). Some jurisdictions have a non-ATAR counting mathematics/numeracy subject ‘below’ Essential Mathematics, and some jurisdictions have advanced university mathematics subjects counting toward the certificate (Victoria, NSW).

There are some minor variations in nomenclature, purpose and status, as summarised in the following table:

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Level | 0 | 1 | 2 | 3 | 4 | 5 |
| Count for ATAR? | No | Maybe | Yes | Yes | Yes | Yes, extension study/advanced standing for university |
| Study type | Survival numeracy  | Everyday life applications | Practical non-calculus; statistics, financial mathematics, business applications, practical geometry | Function, algebra, calculus probability and statistics  | Advanced calculus, differential equations, complex numbers and vectors | University first year |
| State/territory |  |
| ACT | NA | NA | Mathematical Applications | Mathematical Methods | SpecialistMathematics | Yes |
| NSW | NA | Mathematics Life Skills | Mathematics Standard | Mathematics Advanced | MathematicsExtension 1 and Extension 2 | Yes |
| QLD | Short Course in Numeracy (1 unit) | Essential Mathematics | General Mathematics | Mathematical Methods | SpecialistMathematics | No |
| SA/NT | NA | Essential Mathematics | General Mathematics | Mathematical Methods | SpecialistMathematics | No |
| TAS | Everyday Maths | Essential Skills/Workplace- Maths | General Mathematics | Mathematical Methods | SpecialistMathematics | No |
| VIC | NA | Foundation Mathematics (Units 1 and 2 only) | Further Mathematics | Mathematical Methods | SpecialistMathematics | Yes |
| WA | Mathematics Foundation | Mathematics Essential | Mathematics Applications | Mathematics Methods | MathematicsSpecialist  | No |

The following is a summary of senior secondary mathematics studies in various international jurisdictions, comparing the structure and broad content with that of the current Victorian senior secondary curriculum.

Hong Kong structure

The Hong Kong senior secondary mathematics curriculum (2007, updated 2017) has a different structure with a non-calculus compulsory component covering functions and algebra, variation, number including complex numbers, sequences and series, measurement, geometry and trigonometry, and probability and statistics. This is the entry level course for senior secondary students, and in many respects is similar (except for no calculus) to the former Victorian Mathematics A course. Alternatively, it could be regarded as combining aspects of General and Further Mathematics and non-calculus aspects of Mathematical Methods. That is, it is a combination of Australian levels 2 and 3.

For students wishing to do more mathematics, there is an extended curriculum comprising one of two alternative modules: Module 1: Calculus and Statistics or Module 2: Algebra and Calculus.

Module 1 covers calculus content in breadth and depth between Mathematical Methods and Specialist Mathematics (but with no differential equations), probability distribution like Mathematical Methods and statistics from both Mathematical Methods and Specialist Mathematics.

Module 2 covers calculus like Specialist Mathematics (but with no differential equations or kinematics or vector calculus), probability distribution like Mathematical Methods and statistics from both Mathematical Methods and Specialist Mathematics. It also covers mathematical induction, determinants and matrices (2 ×2 or 3 × 3), vectors, but no vector calculus. In broad terms, this structure is like that of the former Victorian Mathematics A and Mathematics B.

These are both combinations of Australian levels 3 and 4 with some exceptions and some additional topics. The 2017 revisions resulted in no change to structure, and mainly comprised refinements of content. There is no material on mechanics in either Compulsory or Extension components, and no courses corresponding to Australian levels 0 or 1.

Singapore structure

The Singapore senior secondary mathematics curriculum is placed within the Sciences group (phased in progressively 2013–2017) and is broadly based on the UK Cambridge O and A level pathways, with O-level and O-level Additional courses (all calculus-based) leading on to H1 and H2 /H2 Further Mathematics courses with a special optional H3 complementary to H2 course for those with specific interests in mathematics. The courses correspond to the Australian levels 2/3 and 4 with aspects of level 5 (H3). These courses are all designed for students progressing to university level studies. There is no provision for Australian level 0–2 courses in this structure.

H1 assumes O-level background and H2/H2 Further Mathematics also assumes O-level Additional background. H1 is aimed at students who require mathematics and statistics to support business or social sciences studies at university. It is an elementary calculus-based course like a blend of the Victorian 1980s General Mathematics/Mathematics A with the regression aspects of the Victorian Further Mathematics Data analysis core, binomial and normal probability distributions and statistical inference related to hypothesis testing for means.

H2 is aimed at students who require mathematics and statistics to support science, engineering and related studies at university. It is like the previous iteration of Specialist Mathematics, without dynamics and content but with statistics material on regression and inference for hypothesis testing for means.

H2 Further Mathematics (from 2016) is designed for mathematically inclined students intending to specialise in mathematics, science and engineering. It extends and expands on H2 and is to be offered in conjunction with it as a double mathematics course. Some of this includes additional content from Victorian Further Mathematics (discrete mathematics recurrence relations), Mathematical Methods (discrete and continuous random variables), Specialist Mathematics (numerical methods, arc length, volume of revolution), as well as other topics such as mathematical induction, polar curves and conic sections, matrices and linear spaces, further hypothesis testing and confidence intervals, and non-parametric tests.

H3 Mathematics (from 2017) assumes H2 and is aimed at students who wish to become mathematicians. It focuses on proof and non-routine problem solving in number, functions, sequences and series, inequalities and counting. Rather than specify mathematical content, it focuses on various processes for working mathematically and a repertoire of mathematical results to be investigated.

A feature of these curricula is the extent to which probability and statistics is embedded in all courses, and the absence of content on mechanics and vector calculus.

Data on enrolment numbers and patterns under this revised structure is not yet generally available.

International Baccalaureate structure (current from 2014–2020 /new 2021–2028)

The IB diploma program offers a liberal education, and its current structure offers four levels of senior secondary mathematics:

[Mathematical studies standard level](https://www.google.com.au/search?biw=1285&bih=929&ei=JyQWW_z0Csqz0gTHwZ-YBA&q=ib+mathematical+studies+syllabus&oq=ib+mathematical+syllabus&gs_l=psy-ab.1.0.0i7i30k1l2j0i13k1l2j0i13i30k1j0i13i5i30k1j0i8i13i30k1l4.720951.723814.0.726887.6.6.0.0.0.0.310.977.2-3j1.4.0....0...1c.1.64.psy-ab..2.4.976...0j0i7i10i30k1.0.VA7mcP__bj8)

This course is like a Victorian level 2/3 Further Mathematics course with elementary calculus aspects of the Victorian 1980s General Mathematics course. It is designed for students who are preparing for a career in social sciences, humanities, languages or arts. These students may need to utilise the statistics and logical reasoning that they have learned as part of the mathematical studies SL course in their future studies. It covers number, algebra, elementary calculus, descriptive statistics and applications, geometry, measurement and trigonometry content, and includes a major project.

[Mathematics standard level (SL)](https://www.google.com.au/search?q=ib+mathematics+sl+syllabus&sa=X&ved=0ahUKEwiwn8bX47vbAhVEo5QKHbDIBMwQ1QIIowIoAA&biw=1285&bih=929)

This course is like an Australian level 3 Mathematical Methods course with some Data analysis material like the Victorian Further Mathematics core and some introductory material on vectors. It is designed for students who will need a sound mathematical background as they prepare for future studies in subjects such as chemistry, economics, psychology and business administration. It covers function, algebra, calculus, statistics and probability content.

[Mathematics higher level (HL)](http://www.sdgj.com/ckfinder/userfiles/files/3cc65c478e92085f8c4d5448ba9a7cd3.pdf)

This course is like an Australian levels 3 and 4 combination of Mathematical Methods and Specialist Mathematics with a majority core and a minor option structure. The core covers function, algebra, calculus, statistics and probability content with options from Statistics and probability, Sets, relations and groups, Calculus and Discrete mathematics (including graph theory). It is designed for students who aim to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology.

[Further mathematics higher level](https://www.spps.org/cms/lib/MN01910242/Centricity/Domain/853/IBFurtherMathGuide2014.pdf)

The course is an Australian level 4/5 course with some aspects of the Victorian Further Mathematics included. It assumes Mathematics HL. It covers content from linear algebra, Euclidean geometry, conic sections and parametric equations, probability and statistics including statistical inference, hypothesis testing, bivariate distributions and covariance, sets, relations and groups and subgroups, infinite series, some advanced calculus, and discrete mathematics (induction, elementary number theory, graphs and networks, and recurrence relations). Students undertaking this course will expect to study mathematics at university, either as a subject in its own right or as a major component of a related subject.

In broad design terms this structure has similarities to both the Australian and Singaporean models; however, all of the courses include calculus (the Studies Standard level course is similar to the Victorian 1980s General Mathematics course) and there is no course corresponding to the Australian Essential Mathematics courses.

The most recent IB review is the outcome of similar work to that of the current VCE Mathematics review in the IB context. The four-level structure has been replaced with a two alternate course structure. In some measure these could roughly be seen as combining the first two and second two of the current four-level courses; however, there is significant review of content and a clear focus on the intentions of the courses. The two new courses have both common and distinctive content based on their orientations.

*Mathematics: Applications and interpretation* – this course is designed for students who enjoy describing the real world and solving practical problems using mathematics, those who are interested in harnessing the power of technology alongside exploring mathematical models and enjoy the more practical side of mathematics.

*Mathematics: Analysis and approaches* – this course is intended for students who wish to pursue studies in mathematics at university or subjects that have a large mathematical content; it is for students who enjoy developing mathematical arguments, problem solving and exploring real and abstract applications, with and without technology.

[**Computer-Based Mathematics**](https://www.computerbasedmath.org/) (CBM) structure

This structure is based on a different paradigm from those discussed above, and those structures and models detailed in the historical paper ‘Senior Secondary mathematics in Victoria: Working Towards Change’*.* Many would be familiar with the background motivations and arguments provided by Conrad Wolfram in his 2010 TED talk, [Teaching kids real math with computers](https://www.ted.com/talks/conrad_wolfram_teaching_kids_real_math_with_computers) and the 2013 YouTube video [Stop Teaching Calculating, Start Learning Maths!](https://www.youtube.com/watch?v=xYONRn3EbYY). The comments for the latter provide a good insight into a range of views on this topic and on mathematics education.

This structure uses a problem-based approach, assuming access to contemporary computational technology. Rather than specify areas of study, topics and content, identify the sorts of problems students are to be able to solve then look at the necessary concepts, understandings and skills. Five broad categories for such activity are:

Data Science (incorporating and extending probability and statistics)

Geometry

Information Theory

Modelling

Architecture of Mathematics

Areas of mathematics or topics studied could include:

Monte Carlo simulation

Significance and risk

Hypothesis creation, testing and interpretation

Model creation and validation techniques

Pattern matching

Graphs and networks

Encryption

Compression and loss

Optimisation

Specification of location and orientation in three dimensions

Digital description of three-dimensional objects

Machine learning

These are developed through an iterative cycle of the following processes:

Define → Translate → Compute → Interpret

The curriculum would be delivered with respect to problem sets introducing mathematical concepts, tools and approaches. These sets could be developed within proposed blocks:

1. Mathematical Modelling and the Tools of Mathematics
2. Mathematics for problems in the physical world
3. Mathematics for problems in the natural world
4. Mathematics for business and life.

A tiered approach devised around common and distinctive problem sets could be employed to provide for levels of mathematical interest, including student selected contexts.

Computation Thinking and Data Science

Two respondents argued for courses that are fundamentally different in nature, rather than incorporation of aspects of their proposals *within* existing courses/subjects, although both provide proposals for doing so. One argues for computational thinking, mathematical reasoning and experimental mathematics as key parts of the curriculum, scaffolded by access to technology. Course structures could be organised in terms of conceptual difficulty, moving through a range of core contexts/applications and related areas/topics of mathematics, which students could then investigate in more depth through projects based on interest. Another outlines Data Science as a *cross-disciplinary* field of study involving statistics, computer science, mathematics and various ‘soft-skills’.

Six broad areas are identified:

1. Data, Gathering, Preparation and Exploration
2. Data Representation and Transformation
3. Computing with Data
4. Data Modelling
5. Data Visualisation and Presentation
6. Science about Data Science

Topics related to these divisions include Problem elicitation, Computing platforms for data science and Using data for predictions. The proposed structure could be the basis for a different component study. It also has natural elements in common with the CBM and Computational Thinking approaches, with module-based project work and with contextualised problem solving. The Data Science approach is based on the investigative cycle:

Framing the problem → gathering, handling and summarising data →

 → analysing data, using models → communication of results

Views

The following discussion is based on views from a range of local, national and international invited respondents that includes mathematics educators, mathematicians, school mathematics coordinators, system mathematics managers and policy analysts, technologists, students and a retired grandfather, informed by discussion with peers and colleagues as applicable, with respect to the question:

What could a suitable senior secondary mathematics curriculum for a liberal democratic society in a developed country for 2020–2030 look like?

The purpose of this discussion is to cover a range of ideas and approaches and stimulate their consideration as part of the consultation process.

(…) observes that school mathematics is often seen predominantly in terms of content (concepts, structures and skills) from domains of mathematics that students learn, and training in related skills to solve various problems from this content, typically assessed through traditional pencil and paper tests. He also notes that mathematics is more than this; it is also a mode of thinking and an intellectual endeavour through which creativity, imagination and flexibility of mind can be developed. He suggests that too much attention to mechanisation focuses on acquisition to the detriment of understanding. For senior secondary mathematics to develop generic skills (problem-solving, critical thinking, reasoning, inquiring, and communication), there needs to be flexibility in the curriculum to cater for student differences, the active use of technology and the use of a broad range of assessment types. Two key questions need to be repeatedly addressed when considering such matters:

* What will the world be like in the next one or two decades?
* What kinds of mathematics learning will be essential for students to be successful in tomorrow’s world?

A curriculum that comprises cores and optional/module components in different combinations offers a possible structure for dealing with this.

(…) from the perspective of an education policy analyst, notes that mathematics and statistics have been used throughout his working life, with the intuitions developed in these areas informing thinking about differing application contexts well beyond the specific computations that may be involved. He comments on application of complex adaptive systems, which are ‘nonlinear, dynamic, recursive, networked, multiple-layered and often stochastic’ and notes that simulation is often a way of dealing with these. However, it is a way of thinking that is very different from how mathematics has traditionally been taught in schools.

(…) comments that while mathematics is arguably the oldest field of human study, it’s still changing fast and this is being substantially driven by the increasing power and ubiquity of digital technologies, leading to the need for a new set of skills for mathematical and statistical thinking carried out in complement with the natural use of technology (techno-mathematical literacy). She notes that mathematics has been taught both for interest and applicability, and gives the following as key ingredients for VCE mathematics:

* the great ideas of mathematics (proof, structure, fascination, beauty)
* the fundamental goal of creative problem-solving and mathematical modelling (prepared to solve non-routine problems with whatever mathematics they have learned)
* capitalise on digital technologies curriculum including using computer algebra system and other software (coding, simulation and contemporary lively approaches to data visualisation and analysis)
* incorporating new areas of application (for example, optional units appropriate to different level courses, biology, AI and machine learning, big data and customer tracking by business)
* changing mathematical emphases, for example algorithmic thinking, use of matrices, simulations and statistics
* a narrative for each course of study, a coherent and connected set of topics and content, a story for each subject that builds the parts into a whole.

(…) points out that there are various kinds of mathematics, engineering, financial, and different logics. In relation to these, the role of practise is to develop understanding, to make mistakes, and then see what happens and generate insight into the processes involved. Students today have different sets of calculation skills; algorithms are important but so also are their verification, that is, a warrant that they do indeed do what they are supposed to do. The twentieth century has seen an increase in the general level of mathematical skills in the general population and this should continue, with some thought on direction, such as more attention to discrete mathematics. Mathematical structures and systems are important, for example linear recurrence relations in financial modelling and networks in social media systems. He also argues that as the curriculum is for the benefit of students, they should have an active role. Rather than being provided with ‘what is good for you’ there should be a clear explanation of what is being taught. Students want to know why they are learning what is being taught, what use is it? Where shall I use it? When shall I use it? Statements of deferred potential applicability are no longer sufficient. This links to previous comments on the notion of narrative, views on generic skills, flexibility and kinds of mathematics learning, as well as challenge to the way things have traditionally been structured in schools. he also comments that there is a virtue in being able to respond to learning techniques ‘just in time’, as well as those learn for ‘just in case’, with the former providing good opportunity for connecting motivation with technique. There is a genuine intellectual effort to learn and become familiar with new techniques on a continual basis as is often required in industry. Technology can be actively used to alleviate pressure on the syllabus and facilitate assessment. The mathematics that is preserved from one time to another is what is required at the time.

Indeed, there is a virtue in being able to ‘forget’ ideas, methods and approaches which may have been significant or useful at a given time, but are subsequently replaced by newer, more general and effective ideas, methods and approaches.

Three students (…) conferred with peers and colleagues to develop their responses.

 Student 1 (aged 18, male, completed VCE in 2018, studied Further Mathematics, Mathematical Methods, Specialist Mathematics, Chemistry, English Language, and Accounting Units 3 and 4 in Year 11) notes that there is a disconnect between the perception of the applicability of mathematics post-schooling and the nature of the teaching and learning process in school. The latter is seen as being content and memory intensive, with a lot of repetition and drill, and skills applied to well-scripted problems in the main, leading to a disconnect between conceptual and procedural difficulty. He identifies the need for teachers to have sound content and pedagogical content knowledge:

 …the two main parts for a maths teacher should be the understanding of the topics they are teaching and equally important … the ability to communicate this knowledge … this is the less observed part of teachers but is important for the mathematical progression of students, as students learn in unique ways which the teacher needs to have the ability to adapt to.

Student 1 argues for a teaching approach that supports conceptual learning with maths topics treated as an inter-connected web of ideas and concepts, to increase students’ ability to attempt questions that may be unfamiliar, while being able to make a reasonable attempt. He relates this to the proficiencies of understanding, fluency, problem solving and reasoning, while noting that technology, as it is widely used outside school, can alleviate the computational load. In short, reduce the content loading, develop conceptual understanding, connections between topics and ideas, and reasoning capability, and make effective use of technology for computation. He also comments on the need for equitable resource provision and challenge for all students, not just those of the ‘chosen’ mathematically able groups. He was subsequently successful in gaining entry to Medicine at Monash University.

Student 2 (aged 24, female, completed VCE in 2012, in the workforce, studied Further Mathematics and Mathematical Method, Literature, Psychology, Legal studies 3/4 in Year 11), comments on the multiplicity of meaning of ‘mathematics’ to different people, and poses the question ‘Once graduated from high school where does maths fit into our worlds and how does this relate to what we were taught in school?’. This student started a Bachelor of Nursing and found the applications of mathematics to be elementary compared to what was studied in school, typically computations done using a pocket calculator. Changing from this course to a Bachelor in Property and Real Estate degree, in particular property valuation, brought a significant change with more involved mathematics being used regularly ‘… all types of valuation techniques have mathematical components at the crux of the determinations’ (recurrence relations, financial modelling, matrices, geometry, data analysis, time series). Technology is also an essential tool for this work, in particular spreadsheets. Student 2 argues for pragmatic connections between broad areas of mathematical study and likely areas of application – indeed that curriculum developers should robustly investigate and incorporate such connections. In this, her views align in part with those of the respondents on Data Science. She notes that Further Mathematics was highly relevant to her chosen field of profession (property valuer) and provides detailed examples of these connections. However, she also notes that there were quite a few students in this class for whom even the content of this study has little connection to their lives post VCE. She uses this observation to argue for the development of a full Unit 1–4 sequence of mathematics that targets (is based on) the application of maths in generalised (everyday) real-life scenarios, effectively the extension of Foundation Mathematics to a full Unit 1–4 sequence. She concludes by quoting [William Thurston](http://www.azquotes.com/author/31394-William_Thurston): ‘Mathematics is not about numbers, equations, computations or algorithms, it is about understanding’.

Student 3 (aged 26, male, completed VCE in 2011, in the workforce with industry-based training and qualification in Software Engineering/Project Management/ Applied Technology fields with a technology company, studied English, Mathematical Methods, IT Applications, Software development and Physics) considers three aspects: delivery of curriculum, curriculum content, and assessment, based on an assumption of active use of technology:

‘The entire curriculum should be delivered through a personal computing device. This includes the theoretical readings, examples, practise questions, in class assessment tasks as well as the final examinations. These will be less distinct, discrete areas as well, as the theory, examples and practise questions will all blend together.’

He envisages that a dynamic digital work environment of a mobile/ tablet kind would be used with touchscreen, freehand input and annotation, and adaptive keyboards being used, as well as audio and camera input. The worksheet area could support multiple tasks, which can be connected, and data (inputs/outputs and the like) updated in real time.

With respect to content he considers that the fundamental mathematics should be covered much as it is currently, with the natural enhancements for learning provided by technology. He sees that for this curriculum cycle the ‘revolution will be in the delivery of the content, while the content will evolve to take advantage of the new technologies’. With respect to new content he considers that we can ‘take a better look at logic and algorithms’ as well as numeric problems that involve data processing. This should be accessible for all students. Student 3 identifies the use of new technology for assessment as one of the key areas for improvement. He sees the correlation between the whole curriculum being delivered by technology and the whole curriculum being assessed by technology, school-based and examinations. He identifies the potential for school-based assessment to be enhanced by provision of a pool of archetypal sample tasks, which are then adapted accordingly by teachers. He also notes that such use of technology would provide the opportunity for more real-time feedback to students progressively throughout their study of VCE Mathematics and could be a source of data that informs subsequent cycles of review.

The grandfather ( …), aged 85, retired, worked in both military and civil aviation in the United Kingdom, United States and Australia on missile systems, military jets (pilot), air traffic control, search and rescue and airport design; involving applications of mathematics related to kinematics and dynamics, geospatial mapping, design geometry, logistics, statistics and financial mathematics, with significant use of instrumentation and computational technologies across various aspects.

He finished secondary schooling in the UK in 1949-50 and studied advanced mathematics. In his response he observes that to be effective the mathematics curriculum needs to be well aligned at a practical level to the requirements of society at that time (in his case initially post-war reconstruction), with real valuing of vocational as well as academic strands.

He identifies two broad categories of economic people in society, the *users* and the *developers* (see also the [Future Skills report](https://www.alphabeta.com/wp-content/uploads/2019/01/google-skills-report.pdf), Google Australia, 2018). Indeed, in his view there are perhaps *too many* students undertaking Australian level 3 courses with respect to the reality of practical alignment with their subsequent work. The users are those who *employ* already established principles, procedures, systems, knowledge and techniques and equipment in their daily work; the developers are those whose jobs are to *investigate* possible futures, such as fusion power, nanotechnology, genetics and the like. He notes that the users category encompasses a large range of activities, from service industry, through trade and to technical applications, so the (mathematics) education system need to be responsive to these, with a main and robust user-based system component. He also notes that for the *developers*, the reach of modern science, including STEM, requires strong mathematics support in data handling, logic, and computers among other things. In his view, developers should be taught practical use of computers in data handling and problem-solving in programming with algorithms, as necessary for complex real-world problem solving, and assessment should incorporate components that reflect this. In broad terms these views align with a two- to four-level system, corresponding to Australian levels 1, 2–3 and 3–4.

(…) provides comments and observations that follow on from the ‘New Directions’(2014) paper with respect to the current context. Firstly, that the current VCE study structure does provide a suitable structure that could be further adapted fit for purpose. He notes that one of the challenges for students in coming to appreciate mathematics is that the ideas are not explained or disentangled from the associated technical procedures. He suggests that technology is helpful when it assists with alleviating this and is directed towards intelligent inquiry and ends; that the continued use of computer algebra systems is desirable, and such technology is pedagogically indispensable for the study of statistics.

He argues for the introduction of coding/algorithms/programming, noting that this would need to be carefully thought out and linked to other content in the curriculum. He also observes that while options provide for flexibility in experimentation with respect to the introduction of new content (for example, the Matrices module in Further Mathematics), the current cycle is one where there may be benefit from consolidation by having units completely prescribed. He provides a range of suggestions with respect to content for the various studies, including explicit mention of proof in Mathematical Methods, and extension of proof in Specialist Mathematics. He notes that the possible removal of the Mechanics area of study would provide opportunity for the introduction of this and other material.

(…) observes that whether mathematics is a compulsory requirement in a senior secondary certificate or not, there is convergence of issues for consideration through accessibility to all, or attractiveness to all, in the cohort. So courses must be designed to appeal to a very wide range of student abilities and motivations either way. She discusses a simple model for considering students’ experiences of the senior secondary mathematics curriculum – they enter with many years study of mathematics already and are either *switched on* or *switched off*. After their senior secondary experiences, they leave in one of the two same states, *switched on* or *switched off.* Each of the four entry–exit combinations has its own set of stories, and she makes the point that we should consider carefully the *overall student experience* before any discussion as to what content and the like is included in the curriculum or not. In essence – what would student reflections be several years afterwards? She observes that a range of factors have combined in the past so that systems tend to develop ‘pure mathematics’ courses for those who are motivated and good at mathematics, and then a course that is ‘more of the same’ to what they have already done for those who are less motivated or able at mathematics. It is not enough for students to be able to perform various mathematical tasks (instrumental understanding), they also need to appreciate why each of the ideas and relationships involved work the way they do (relational understanding). With respect to the use of technology, she believes students should be able to use any technology they feel is appropriate and the curriculum should define minimum requirements.

She notes that there is a large degree of commonality in the content across systems around the world and relates this to the tendency to construct curricula for those students preparing for future mathematical studies - within such an approach there is little need for change in content, and indeed many current courses are not significantly different in content to those from decades ago. Trigonometry, geometry, functions, calculus, sequences, vectors are present in the majority of systems; any changes over the years have been modest. Mechanics has much importance in the UK, but rarely elsewhere, often being absent in other systems. Decision mathematics, linear programming and the study of algorithms seem to be of increasing importance. While mathematical modelling appears in most curricula, this is often not aligned with adequate pedagogical treatment or suitable assessment structures. This is a consideration raised by several respondents and noted as being in some tension with skill- acquisition, time constraints and the coverage of content perspective.

She thinks that content should be arranged by concept and linked throughout with reference to international contexts, the beauty of mathematics, using technology and promoting ethical and moral judgements. The main themes for thinking about content could be functions, movement and position, statistical literacy and calculus.

(…) observes that most of the population has no *direct* need for much of the content currently taught in senior secondary mathematics. The purpose and motivation for teaching mathematics is in relation to developing reasoning capacity that can be applied in many areas and endeavours. He argues, similarly to several others, that by hand and technology-based approaches need to be carefully designed to be complementary, the effective use of technology flows on from an intrinsic understanding of concepts, skills and processes that are learned by first ‘doing it yourself’. Technology should be used in circumstances where it clearly enhances learning or is necessarily used to relieve computational load.

He proposes a three-tier structure across Units 1–4, a general non-calculus-based study aimed at the broad population, and two calculus-based studies in pure mathematics, with Tier 3, an advanced study aimed at the cohort of able and mathematically inclined students. In some ways this corresponds to the ‘users’ and ‘developers’ in the grandfather response, as well as the revised IB two study structure. Like the grandfather, he notes that there are likely too many students undertaking the Tier 2 study when they would be more suitably catered for in the Tier 1 study. The Tier 2 study would be of a similar nature intended for a larger cohort seeking to study in fields with strong quantitative discipline requirements. The Tier 2 study would be a pre- or co-requisite for the Tier 3 study. Proof should be strongly represented in the Tiers 2 and 3 courses. Although not explicitly stated, it seems that Tier 1 and Level 2 studies would be alternative to each other:

|  |
| --- |
| **Tier 1 study** |
| **Units 1 and 2** | **Units 3 and 4** |
| Practical and financial arithmetic | Data analysis |
| Linear equations and models | Optimisation |
| Measurement | Networks |
| Probability and statistics | Statistical inference |
| Fermi estimation | Algorithms |

|  |
| --- |
| **Tier 2 study** |
| **Units 1 and 2** | **Units 3 and 4** |
| Set theory and logic | Calculus |
| Number systems | Probability |
| Functions |  |
| Calculus |  |
| **Tier 3 study** |
| **Units 1 and 2** | **Units 3 and 4** |
| Proof techniques | Advanced calculus |
| Sequences and series | Polynomials |
| Number theory | Complex numbers |
| Vector and matrices | Vector and matrices |
| Combinatorics | Groups |
| Graph theory | Mechanics |

He notes that, given the increased focus on algorithms and computational thinking for the relationship between the Mathematics curriculum and the Computer science curriculum, it would be useful to explore more closely their natural complementary aspects.

(…) uses the following guiding principles - that a proposal for the senior secondary mathematics curriculum for 2020–2030 should:

* challenge the process and move beyond the status quo
* be ambitious
* be flexible, adaptable and dynamic
* be designed and brought to life by a diverse collective of experts and practitioners
* have the student and their learning at its heart

She argues that curriculum and related syllabuses need to be dynamic and flexible:

…rather than dictating and controlling all aspects of the learning and assessment landscape, they should provide sets of guiding boundaries in a way that allows (or provides clear permission) and encourages schools and teachers to: experiment and try different things; transport the content from the classroom into the real world; nurture curiosity, inquisitiveness and creativity.

She notes that the way we were taught has for many teachers defined and defines the way they teach; however, what was fit for purpose 30 years ago, or even more recently, may no longer be best practice. She observes that many of our students are not learning the mathematics they are expected to know, want to know or need to know. While areas of study such as geometry, algebra and calculus have been the cornerstones of mathematics and their place and importance in the curriculum is ‘ferociously protected and pursued’, such areas need to be re-calibrated to allow for greater inclusion of other areas of mathematics that might more aptly address current and future needs, such as: statistics, probability, discrete mathematics, applied mathematics, logic and set theory and boolean algebra.

With respect to technology, she argues that the full set of available tools and technologies should be used; however, it is critical that the pedagogical content knowledge of teachers with respect to their use is well developed and aligned with the curriculum so that such use goes beyond what could be achieved without it.

She observes that one approach would be to create additional specialised courses of study of the kind proposed by the Data Science proposers or the Wolfram teams; however, such an approach could be problematic, with respect to the connection between current F–10 curricula and the senior secondary years.

She proposes a four-course structure while noting that two advanced courses could be combined in a single higher-level study like the AC Specialist Mathematics, summarised in the following table.

Each course would be a two-year (four unit) sequence, covering the following domains:

**Course 1** – for students seeking to develop mathematical skills directly transferable to everyday contexts:

probability and statistics

measurement and design

finance and business

networks

matrices.

**Course 2** – for students preparing for further study in areas requiring strong mathematical background:

calculus

circular, exponential and logarithmic functions

sequences and series

finance

data analytics

matrices

probability and statistics.

**Course 3** – extension beyond Course 2:

further functions and calculus

predictive analytics

vectors and mechanics

set theory and logic

proof.

**Course 4** – further extension beyond Course 3:

complex numbers

differential equations

vectors, mechanics and design

further set theory and logic

boolean algebra.

In addition, she proposes a 60-hour mathematics project course, which could encompass suggestions such of those made by a range of other respondents.

(…) describe what they ideally would see as a distinctive Data Science study in its own right, as a *cross-disciplinary* field of study involving statistics, computer science, mathematics and various ‘soft-skills’. Such a proposal would be an *alternative* to the current mathematics study offerings and resonates with views of Fadel-CCR and Wolfram Research and teams. They propose inclusion within the Mathematics and Computer Science curriculum modules that cover:

* gathering, handling and summarising data
* computing with data and modelling data for a range of problems and data types
* effective communication for problem solving with data

and strongly recommend that these modules integrate material to be presented as [Data Science](https://courses.csail.mit.edu/18.337/2015/docs/50YearsDataScience.pdf) rather than a collection of loosely associated topics.

Conrad Wolfram and team put forward the fundamental proposition that the historical mathematics curriculum in place for say the last half century is not a suitable model for future progress, as it is premised on the concept of by hand human computation which is both outmoded, inefficient and not relevant to the conceptual complexity of problems at hand today. The structure of a possible alternative approach has been outlined in the previous section. They argue for regularly re-considering the key questions of ‘what are today’s human survival skills?’ and ‘what are the top human value-adds?’. Indeed, the broad applicability of mathematics across a range of fields substantially arises because of the increased computational capability through technology. They observe that much use of technology in education is based around supporting the learning of by hand manipulative procedures of the current curriculum, which is the reverse of real-life application where the compute is the tool for computation and humans instruct what computations should be done.

There is still a place for by hand computation, but it should be aimed at developing underpinning conceptual understanding, as other respondents note, without practise there can be no understanding; however, practise has changed its meaning. With practise comes mistakes, hence learning and understanding of the limitations of computation and being able to detect errors. There will be both different sets of calculating skills and purposes. Students should be able to show that an algorithm does what it is intended to do, leading to verification and proof.

Conrad Wolfram and team comment that today’s curricula seem to work well for only a small fraction of the population. They suggest that the approach they propose is more likely to enfranchise a broader group, since they will not be ‘filtered out’ by lack of technical rather than conceptual capacity, and are more likely to be engaged by starting with a problem and then using abstraction so that mathematical computation can be applied to that problem.

Stephen Wolfram and team articulate several views on the nature of computation, mathematics, thinking in these areas and the relationship between them. They observe that access to computational tools naturally stimulates the use of mathematics and mathematical thinking, for example, the notion of a function, and that a key aspect of experimental mathematics is to look for interesting ‘facts’, those that relate to things we care about. These then connect to larger frameworks and the traditions of mathematics. In tackling such questions students become aware of the maths they need to know as they need to know it.

One of the key ideas is that of a [computational essay](http://blog.stephenwolfram.com/2017/11/what-is-a-computational-essay/), an interactive document where discussion, argument, computations, data and analysis are all combined to tackle a problem or investigate an area of interest. Indeed, such [investigations and projects](http://blog.stephenwolfram.com/2017/08/high-school-summer-camp-a-two-week-path-to-computational-thinking/#more-13854) can be practical or a vehicle for deeper study of mathematics and mathematical questions.

Charles Fadel and colleagues at the Center for Curriculum Redesign (CCR) presented this paper at the [Maths for the 21st Century conference](http://curriculumredesign.org/maths-for-the-21st-century-conference-strongly-supports-critical-necessary-changes-of-oecds-pisa-maths-for-2021-4dedu/) on 25 May 2018, and forwarded an earlier draft as response for the present work. Other presenters included Keith Devlin and Conrad Wolfram. Fadel’s paper ‘PISA Mathematics in 2021’ articulates recommendations for a re-developed framework for mathematical literacy for PISA. While the paper does not specifically provide a structure or model for implementation of senior secondary curriculum, it discusses design elements and argues for the continued importance of Mathematics in the curriculum, provides a rationale for proposed improvements, and covers the following areas: explicit reasoning and process, knowledge relevance, developing competencies and innovative tools.

PISA assessments are for students aged 15 years old, that is, just prior to senior secondary mathematics in their schooling. They are based on the construct of mathematical literacy defined as: ‘An individual’s capacity to identify and understand the role that mathematics play in the world, to make well-founded judgements and to use and engage with mathematics in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen.’ The paper notes that for OECD countries mathematics accounts on average for 15 per cent of total instructional time in the curriculum at this level. At the VCE level, a single mathematics study would typically account for 16 – 17 per cent of the total instructional time at Units 1 and 2 levels; and 20 per cent of the total instructional time at Units 3 and 4 levels.

The paper notes that modern industry requires different mathematics above and beyond the traditional branches of arithmetic, geometry and algebra, as indicated in the following table:



Similar to the work of Conrad Wolfram and team, the PISA assessments focus on three mathematical processes:

*formulating* situations mathematically

*employing* mathematical concepts, facts, procedures, and reasoning

interpreting, applying and evaluating mathematical outcomes

and relate these to seven reasoning tools:



...which are to be applied across four main areas of mathematics:

*shape and space*: unpredictable shapes, proportionality

*change and relationships*: exponential functions, proportionality, algorithmic mathematics

*uncertainty and data*: Bayesian/conditional probability, discrete maths (combinatorics, game theory, complex systems)

*quantity*: number sense, estimation.

The paper notes that the design challenge is to:

* increase focus on important existing areas for content
* identify and de-emphasise streamline/remove some existing content
* incorporate new important/relevant areas and content.

These need to be balanced with respect to the instructional time available and the use of technology. The paper also defines the following progression in mathematical creativity and metacognition:

* solve exercises and problems using standard solutions
* solve exercises and problems using non-standard solutions (creative stretch)
* find new real-world problems, and solve using both standard and non-standard solutions (creative stretch)
* create new problems, and solve using both standard and non-standard solutions (creative stretch)
* create new classes of problems (metacognitive stretch) and explore solvability.