VCE Specialist Mathematics Unit 1

Mathematical investigation 1: Squares of integers

Formulation

The squares of integers are involved in a lot of questions in number theory. There are numerical investigations to undertake to help make hypotheses and proofs to be undertaken to establish results.

Exploration

A Sums of squares and higher powers

1. Prove that *k*3 – (*k* – 1)3 = 3*k*2 – 3*k* +1.

Thus $\sum\_{k=1}^{n}k^{3} – (k – 1)^{3}= \sum\_{k=1}^{n}3k^{2}– 3k +1.$ First explain why the LHS of this equation is equal to *n*3. Use this result to prove

$\sum\_{k=1}^{n}k^{2}= \frac{n(n+1)(2n+1)}{6}$.

Now we have the result also prove the result by induction.

1. Use a similar technique to find $\sum\_{k=1}^{n}k^{3}$ and discuss how to extend to $\sum\_{k=1}^{n}k^{a}$ for any positive integer *n*.
2. Find expressions for the sum of the first *n* odd numbers and the first *n* even numbers.
3. Write pseudocode to find $\sum\_{k=1}^{n}\frac{1}{k^{2}}$ for a given *n*. Use a device to run your program for
*n* = 10, 20, 30, …,100. Comment on the results.
4. Write pseudocode to find $\sum\_{k=1}^{n}\frac{1}{k^{3}}$ for a given *n*. Use a device to run your program for
*n* = 10, 20, 30, …,100. Comment on the results.

B Differences of squares

1. That any odd number can be written as the difference of two squares.
2. Prove that all numbers of the form 4*k*, where *k*is a non-negative integer, can be written as the difference of two squares.
3. Prove that no number of the form 4*k* + 2, where k is a non-negative integer, can be written as the difference of two squares.
4. Prove that any number of the form *pq*, where *p* and *q* are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways.
5. Consider some numerical examples to illustrate these results. Have the above results covered all positive integers?

C Sums of squares

1. Develop an algorithm using pseudocode to find all the numbers less than 101 that are the sum of two squares. Use a device to utilise this algorithm. Print (*a,b,c*) where *c* = *a*2 + *b*2. Check if a given representation of *c* is unique.

Carefully consider cases where *c* is prime. Also consider the results where *a* and *b* are prime.

1. With some messy algebra show that if a number is the sum of two non-zero squares, then the square of that number is also the sum of two non-zero squares.

Also show that the product of such numbers is also the sum of two square numbers.

See [Euler’s proofs on numbers which are the sum of two squares](http://eulerarchive.maa.org/docs/translations/E228en.pdf).

Conclusions

Summarise your findings and illustrate with examples. Discuss any weaknesses with your algorithms and comment how they could be improved and possibly generalised. Summarise the algebraic techniques and techniques of proof that you have used. You may like to add results.

Areas of study

The following content from the areas of study is addressed through this learning activity.

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| --- | --- | --- |
| **Area of study** | **Topic** | **Content dot point** |
| Algebra, number and structure | Proof and number | 1, 4, 5, 6 |

Outcomes

The following outcomes, key knowledge and key skills are addressed through this task.

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| --- | --- | --- |
| **Outcome** | **Key knowledge dot point** | **Key skill dot point** |
| 1 | 1, 3, 14, 15 | 1, 2, 14 |
| 2 | 1, 2, 3, 4, 5 | 1, 2, 3, 4, 6 |
| 3 | 1, 2, 6 | 1, 2, 4, 7, 10, 11, 12 |