VICTORIAN CURRICULUM
AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2023

Print exam correction: Section B, Q6h., diagram labels have been switched; Formula Sheet: page 2., Data analysis, probability and statistics table, 4th box, '+' has been added before X_n

STUDENT NUMBER | | | | | | | | | | Letter | |

SPECIALIST MATHEMATICS

Written examination 2

Monday 6 November 2023

Reading time: 11.45 am to 12.00 pm (15 minutes)
Writing time: 12.00 pm to 2.00 pm (2 hours)

QUESTION AND ANSWER BOOK

Structure of book

Section	*Number of questions*	*Number of questions to be answered*	*Number of marks*
A	20	20	20
B	6	6	60
			Total 80

- Students are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set squares, aids for curve sketching, one bound reference, one approved technology (calculator or software) and, if desired, one scientific calculator. Calculator memory DOES NOT need to be cleared. For approved computer-based CAS, full functionality may be used.
- Students are NOT permitted to bring into the examination room: blank sheets of paper and/or correction fluid/tape.

Materials supplied
- Question and answer book of 23 pages
- Formula sheet
- Answer sheet for multiple-choice questions

Instructions
- Write your **student number** in the space provided above on this page.
- Check that your **name** and **student number** as printed on your answer sheet for multiple-choice questions are correct, **and** sign your name in the space provided to verify this.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
- All written responses must be in English.

At the end of the examination
- Place the answer sheet for multiple-choice questions inside the front cover of this book.
- You may keep the formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

SECTION A – Multiple-choice questions

Instructions for Section A

Answer **all** questions in pencil on the answer sheet provided for multiple-choice questions.
Choose the response that is **correct** for the question.
A correct answer scores 1; an incorrect answer scores 0.
Marks will **not** be deducted for incorrect answers.
No marks will be given if more than one answer is completed for any question.
Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1

Consider the following statement.
'If my football team plays badly, then they are not training enough.'
Which one of the following statements is the contrapositive of the statement above?

A. If they are not training enough, then my football team plays badly.
B. If my football team plays badly, then they need more training.
C. If they are training enough, then my football team does not play badly.
D. If my football team doesn't play badly, then they are training enough.
E. If they are training enough, then my football team will most likely win.

Question 2

The graph of $y = \dfrac{x^3}{ax^2 + bx + c}$ has asymptotes given by $y = 2x + 1$ and $x = 1$. The values of a, b and c are, respectively

A. $2, -4, 2$
B. $\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}$
C. $\frac{1}{2}, \frac{1}{4}, -\frac{3}{4}$
D. $\frac{1}{2}, -\frac{1}{4}, -\frac{3}{4}$
E. $2, -4, -8$

Question 3

In the interval $-\pi \le x \le \pi$, the graph of $y = a + \sec(x)$, where $a \in R$, has two x-intercepts when

A. $0 \le a \le 1$
B. $-1 < a < 1$
C. $a \le -1$ or $a > 1$
D. $-1 \le a < 0$
E. $a < -1$ or $a \ge 1$

DO NOT WRITE IN THIS AREA

Question 4

If $z = -(2a+1) + 2ai$, where a is a non-zero real constant, then $\dfrac{4a}{1+\overline{z}}$ is equal to

A. $\sqrt{2}\text{cis}\left(\dfrac{\pi}{4}\right)$

B. $\sqrt{2}\text{cis}\left(\dfrac{3\pi}{4}\right)$

C. $\text{cis}\left(\dfrac{\pi}{4}\right)$

D. $\sqrt{2}\text{cis}\left(-\dfrac{3\pi}{4}\right)$

E. $\text{cis}\left(-\dfrac{\pi}{4}\right)$

Question 5

Let z be a complex number where $\text{Re}(z) > 0$ and $\text{Im}(z) > 0$.

Given $|\overline{z}| = 4$ and $\arg(z^3) = -\pi$, then z^2 is equivalent to

A. $4z$

B. $-2\overline{z}$

C. $3z$

D. $\overline{z}^2$

E. $-4\overline{z}$

Question 6

Consider the following pseudocode.

```
define f(x,y)=e^(xy)
   x←0
   y←0
   h←0.5
   n←0
while n≥0
   y←y+h×f(x,y)
   x←x+h
   n←n+1
print y
end while
```

After how many iterations will the pseudocode print 2.709?

A. 1

B. 2

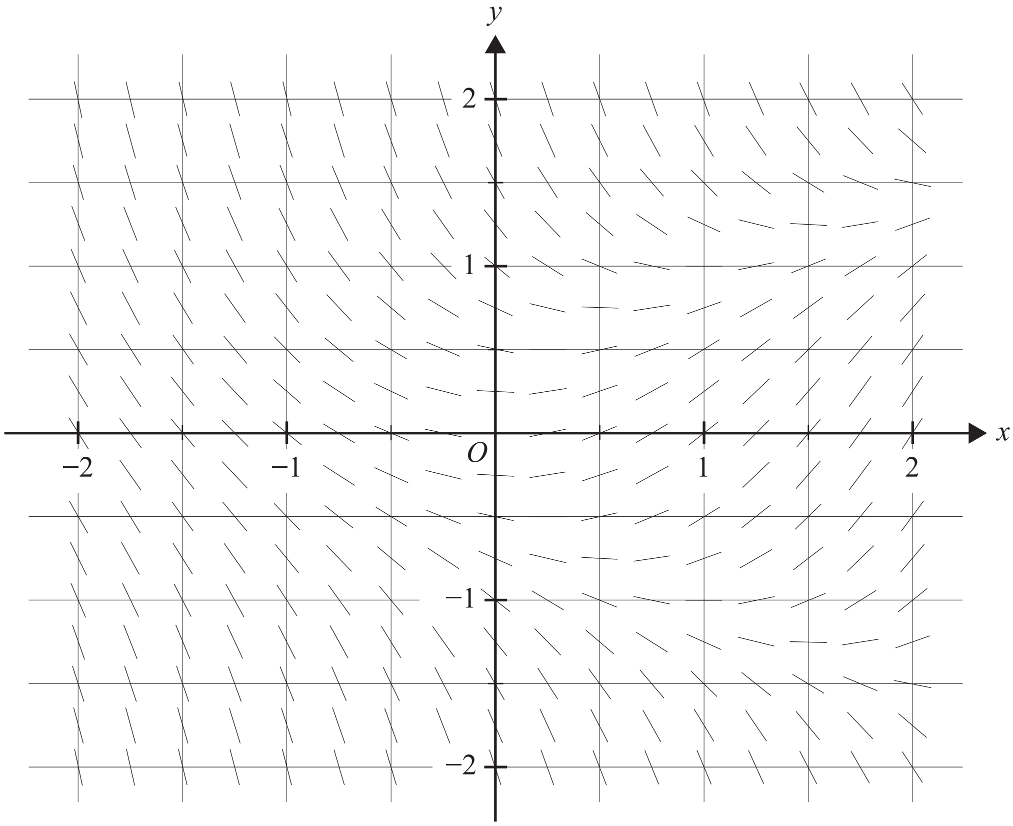
C. 3

D. 4

E. 5

DO NOT WRITE IN THIS AREA

Question 7



The direction field for a differential equation is shown above. On a certain solution curve of this differential equation, $y = 2$ when $x = -1$.

The value of y on the same solution curve when $x = 1.5$ is closest to

A. -0.5

B. 0

C. 0.5

D. 1.0

E. 1.5

DO NOT WRITE IN THIS AREA

Question 8

Initially a spa pool is filled with 8000 litres of water that contains a quantity of dissolved chemical. It is discovered that too much chemical is contained in the spa pool water. To correct this situation, 20 litres of well-mixed spa pool water is pumped out every minute while 15 litres of fresh water is pumped in each minute.

Let Q be the number of kilograms of chemical that remains dissolved in the spa pool after t minutes.

The differential equation relating Q to t is

A. $\frac{dQ}{dt} = \frac{4Q}{t-1600}$

B. $\frac{dQ}{dt} = \frac{-Q}{400}$

C. $\frac{dQ}{dt} = \frac{3Q}{t-1600}$

D. $\frac{dQ}{dt} = \frac{3Q}{1600-t}$

E. $\frac{dQ}{dt} = \frac{4Q}{1600-t}$

Question 9

The position of a particle moving in the Cartesian plane, at time t, is given by the parametric equations

$x(t) = \frac{6t}{t+1}$ and $y(t) = \frac{-8}{t^2+4}$, where $t \geq 0$.

What is the slope of the tangent to the path of the particle when $t = 2$?

A. $-\frac{1}{3}$

B. $-\frac{1}{4}$

C. $\frac{1}{3}$

D. $\frac{3}{4}$

E. $\frac{4}{3}$

Question 10

If $I_n = \int_0^1 \left((1-x)^n e^x\right) dx$, where $n \in N$, then for $n \geq 1$, I_n equals

A. $-1 + nI_{n-1}$

B. nI_{n-1}

C. $-1 - nI_{n-1}$

D. $-nI_{n-1}$

E. $(1-x)^n e^x + nI_{n-1}$

Question 11

The area of the curved surface generated by revolving part of the curve with equation $y = \cos^{-1}(x)$ from $\left(0, \frac{\pi}{2}\right)$ to $(1, 0)$ about the y-axis can be found by evaluating

A. $2\pi \int_0^{\frac{\pi}{2}} \left(\cos^{-1}(x) \sqrt{1 + \frac{1}{x^2 - 1}} \right) dx$

B. $2\pi \int_0^{1} \left(\cos^{-1}(x) \sqrt{1 + \frac{1}{x^2 - 1}} \right) dx$

C. $2\pi \int_0^{\frac{\pi}{2}} \cos(y) \sqrt{1 - \sin^2(y)}\, dy$

D. $2\pi \int_0^{\frac{\pi}{2}} \sqrt{1 + u^2}\, du$, where $u = \sin(y)$

E. $2\pi \int_0^{1} \sqrt{1 + u^2}\, du$, where $u = \sin(y)$

Question 12

The acceleration, a m s^{-2}, of a particle that starts from rest and moves in a straight line is described by $a = 1 + v$, where v m s^{-1} is its velocity after t seconds.

The velocity of the particle after $\log_e(e + 1)$ seconds is

A. e

B. $e + 1$

C. $e^2 + 1$

D. $\log_e(1 + e) + 1$

E. $\log_e\left(\log_e(1 + e) - 1\right)$

Question 13

A tourist in a hot air balloon, which is rising vertically at 2.5 m s^{-1}, accidentally drops a phone over the side when the phone is 80 metres above the ground.

Assuming air resistance is negligible, how long in seconds, correct to two decimal places, does it take for the phone to hit the ground?

A. 2.86

B. 2.98

C. 3.79

D. 4.04

E. 4.30

Question 14

Let $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$.

If $\underset{\sim}{n}$ is a unit vector such that $\underset{\sim}{a}.\underset{\sim}{n} = 0$ and $\underset{\sim}{b}.\underset{\sim}{n} = 0$, then $|\underset{\sim}{c}.\underset{\sim}{n}|$ is equal to

A. 2

B. 3

C. 4

D. 5

E. 6

Question 15

If the sum of two unit vectors is a unit vector, then the magnitude of the difference of the two vectors is

A. 0

B. $\frac{1}{\sqrt{2}}$

C. $\sqrt{2}$

D. $\sqrt{3}$

E. $\sqrt{5}$

Question 16

A student throws a ball for his dog to retrieve. The position vector of the ball, relative to an origin O at ground level t seconds after release, is given by $\underset{\sim}{r}_B(t) = 5t\underset{\sim}{i} + 7t\underset{\sim}{j} + (15t - 4.9t^2 + 1.5)\underset{\sim}{k}$. Displacement components are measured in metres, where $\underset{\sim}{i}$ is a unit vector to the east, $\underset{\sim}{j}$ is a unit vector to the north and $\underset{\sim}{k}$ is a unit vector vertically up.

The total **vertical** distance, in metres, travelled by the ball before it hits the ground is closest to

A. 1.5

B. 11.5

C. 13.0

D. 24.5

E. 26.0

Question 17

Consider the vectors $\underset{\sim}{a} = \alpha\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 3\underset{\sim}{i} + \beta\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - 7\underset{\sim}{j} + \gamma\underset{\sim}{k}$, where $\alpha, \beta, \gamma \in R$. If $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{c}$, then

A. $\alpha = -2, \beta = -1, \gamma = -5$

B. $\alpha = -1, \beta = 2, \gamma = -1$

C. $\alpha = 1, \beta = -2, \gamma = -5$

D. $\alpha = -2, \beta = -1, \gamma = -1$

E. $\alpha = 1, \beta = -2, \gamma = 5$

Question 18

What value of k, where $k \in R$, will make the following planes perpendicular?

$$\Pi_1 : 2x - ky + 3z = 1$$
$$\Pi_2 : 2kx + 3y - 2z = 4$$

A. 2

B. 4

C. 6

D. 8

E. 10

Question 19

A company accountant knows that the amount owed on any individual unpaid invoice is normally distributed with a mean of \$800 and a standard deviation of \$200.

What is the probability, correct to three decimal places, that in a random sample of 16 unpaid invoices the **total** amount owed is more than \$13 500?

A. 0.087

B. 0.191

C. 0.413

D. 0.587

E. 0.809

Question 20

The lifespan of a certain electronic component is normally distributed with a mean of μ hours and a standard deviation of σ hours.

Given that a 99% confidence interval, based on a random sample of 100 such components, is (10 500, 15 500), the value of σ is closest to

A. 9710

B. 10 750

C. 12 750

D. 15 190

E. 19 390

END OF SECTION A

DO NOT WRITE IN THIS AREA

SECTION B

Instructions for Section B

Answer **all** questions in the spaces provided.

Unless otherwise specified, an **exact** answer is required to a question.

In questions where more than one mark is available, appropriate working **must** be shown.

Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

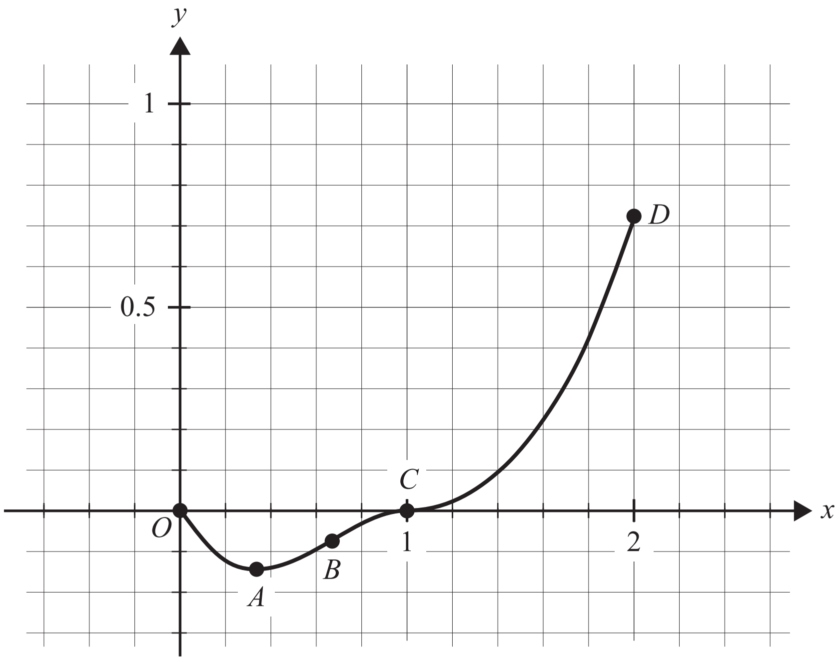
Take the **acceleration due to gravity** to have magnitude g m s^{-2}, where $g = 9.8$

Question 1 (10 marks)

Viewed from above, a scenic walking track from point O to point D is shown below. Its shape is given by

$$f(x) = \begin{cases} -x(x+a)^2, & 0 \le x \le 1 \\ e^{x-1} - x + b, & 1 < x \le 2\,. \end{cases}$$

The minimum turning point of section $OABC$ occurs at point A. Point B is a point of inflection and the curves meet at point $C(1, 0)$. Distances are measured in kilometres.



a. Show that $a = -1$ and $b = 0$. 1 mark

b. Verify that the two curves meet **smoothly** at point C. 2 marks

c. **i.** Find the coordinates of point A. 1 mark

ii. Find the coordinates of point B. 1 mark

The return track from point D to point O follows an elliptical path given by

$$x = 2\cos(t) + 2,\ y = (e - 2)\sin(t), \text{ where } t \in \left[\frac{\pi}{2},\ \pi\right].$$

d. Find the Cartesian equation of the elliptical path. 2 marks

e. Sketch the elliptical path from D to O on the diagram on page 10. 1 mark

f. **i.** Write down a definite integral in terms of t that gives the length of the elliptical path from D to O. 1 mark

ii. Find the length of the elliptical path from D to O.
Give your answer in kilometres correct to three decimal places. 1 mark

DO NOT WRITE IN THIS AREA

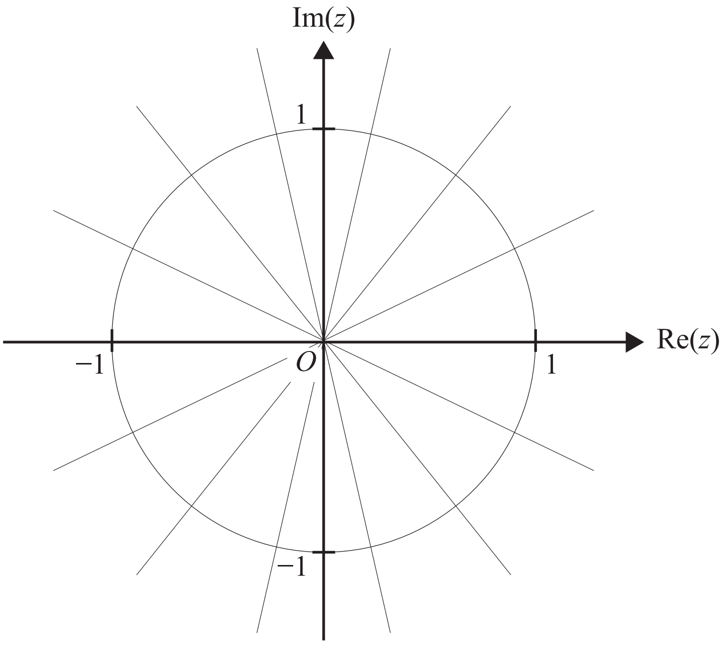
Question 2 (10 marks)

Let $w = \text{cis}\left(\dfrac{2\pi}{7}\right)$.

a. Verify that w is a root of $z^7 - 1 = 0$. 1 mark

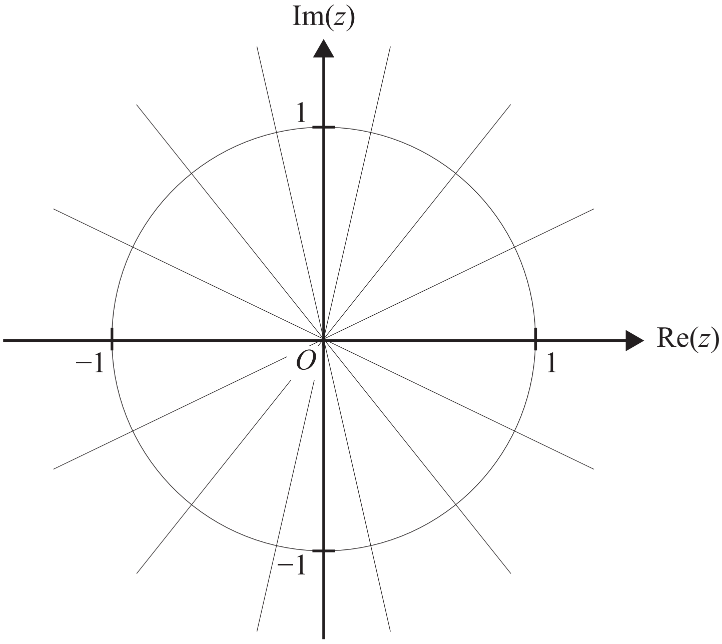
b. List the other roots of $z^7 - 1 = 0$ in polar form. 1 mark

c. On the Argand diagram below, plot and label the points that represent all the roots of $z^7 - 1 = 0$. 2 marks



DO NOT WRITE IN THIS AREA

d. **i.** On the Argand diagram below, sketch the ray that originates at the real root of $z^7 - 1 = 0$ and passes through the point represented by $\text{cis}\left(\frac{2\pi}{7}\right)$. 1 mark



ii. Find the equation of this ray in the form $\text{Arg}(z - z_0) = \theta$, where $z_0 \in C$, and θ is measured in radians in terms of π. 1 mark

e. Verify that the equation $z^7 - 1 = 0$ can be expressed in the form $(z - 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$. 1 mark

f. **i.** Express $\text{cis}\left(\frac{2\pi}{7}\right) + \text{cis}\left(\frac{12\pi}{7}\right)$ in the form $A\cos(B\pi)$, where $A, B \in R^+$. 1 mark

ii. Given that $w = \text{cis}\left(\frac{2\pi}{7}\right)$ satisfies $(z - 1)(z^6 + z^5 + z^4 + z^3 + z^2 + z + 1) = 0$, use De Moivre's theorem to show that $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$. 2 marks

DO NOT WRITE IN THIS AREA

Question 3 (10 marks)

The curve given by $y^2 = x - 1$, where $2 \le x \le 5$, is rotated about the x-axis to form a solid of revolution.

a. **i.** Write down the definite integral, in terms of x, for the volume of this solid of revolution. 1 mark

ii. Find the volume of the solid of revolution. 1 mark

b. **i.** Express the curved surface area of the solid in the form $\pi \int_a^b \sqrt{Ax - B}\, dx$, where a, b, A, B are all positive integers. 2 marks

ii. Hence or otherwise, find the curved surface area of the solid correct to three decimal places. 1 mark

The total surface area of the solid consists of the curved surface area plus the areas of the two circular discs at each end.

The 'efficiency ratio' of a body is defined as its total surface area divided by the enclosed volume.

c. Find the efficiency ratio of the solid of revolution correct to two decimal places. 2 marks

DO NOT WRITE IN THIS AREA

d. Another solid of revolution is formed by rotating the curve given by $y^2 = x - 1$ about the x-axis for $2 \le x \le k$, where $k \in R$. This solid has a volume of 24π.

Find the efficiency ratio for this solid, giving your answer correct to two decimal places. 3 marks

DO NOT WRITE IN THIS AREA

Question 4 (10 marks)

A fish farmer releases 200 fish into a pond that originally contained no fish. The fish population, P, grows according to the logistic model, $\frac{dP}{dt} = P\left(1 - \frac{P}{1000}\right)$, where t is the time in years after the release of the 200 fish.

a. The above logistic differential equation can be expressed as

$$\int \frac{A}{P} + \frac{B}{1 - \frac{P}{1000}} dP = \int dt, \text{ where } A, B \in R.$$

Find the values of A and B. 1 mark

One form of the solution for P is $P = \frac{1000}{1 + De^{-t}}$, where D is a real constant.

b. Find the value of D. 1 mark

The farmer releases a batch of n fish into a second pond, pond 2, which originally contained no fish. The population, Q, of fish in pond 2 can be modelled by $Q = \frac{1000}{1 + 9e^{-1.1t}}$, where t is the time in years after the n fish are released.

c. Find the value of n. 1 mark

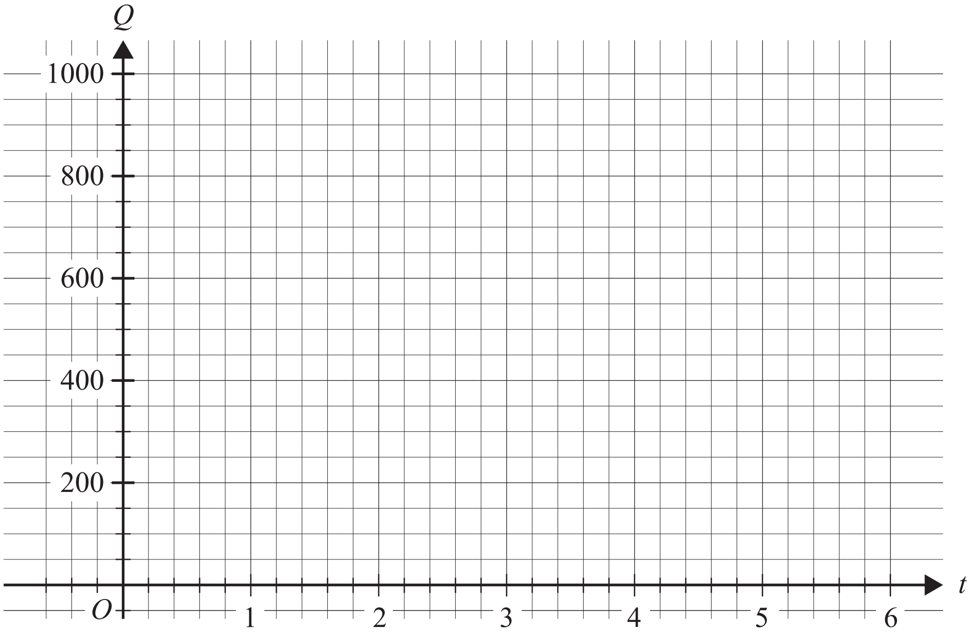
d. Find the value of Q when $t = 6$.

Give your answer correct to the nearest integer. 1 mark

e. **i.** Given that $\frac{dQ}{dt} = \frac{11}{10}Q\left(1 - \frac{Q}{1000}\right)$, express $\frac{d^2Q}{dt^2}$ in terms of Q. 1 mark

ii. Hence or otherwise, find the size of the fish population in pond 2 and the value of t when the rate of growth of the population is a maximum. Give your answer for t correct to the nearest year. 2 marks

f. Sketch the graph of Q versus t on the set of axes below. Label any axis intercepts and any asymptotes with their equations. 2 marks



The farmer wishes to take 5.5% of the fish from pond 2 each year. The modified logistic differential equation that would model the fish population, Q, in pond 2 after t years in this situation is

$$\frac{dQ}{dt} = \frac{11}{10}Q\left(1 - \frac{Q}{1000}\right) - 0.055Q\,.$$

g. Find the maximum number of fish that could be supported in pond 2 in this situation. 1 mark

Question 5 (11 marks)

The points with coordinates $A(1, 1, 2)$, $B(1, 2, 3)$ and $C(3, 2, 4)$ all lie in a plane Π.

a. Find the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and hence show that the area of triangle ABC is 1.5 square units. 2 marks

b. Find the shortest distance from point B to the line segment AC. 2 marks

A second plane, ψ, has the Cartesian equation $2x - 2y - z = -18$.

c. At what acute angle does the line given by $\underset{\sim}{r}(t) = 3\underset{\sim}{i} + 2\underset{\sim}{j} + 4\underset{\sim}{k} + t\left(\underset{\sim}{i} - 2\underset{\sim}{j} + 2\underset{\sim}{k}\right),\ t \in R$, intersect the plane ψ? Give your answer in degrees correct to the nearest degree. 2 marks

A line L passes through the origin and is normal to the plane ψ. The line L intersects ψ at a point D.

d. Write down an equation of the line L in parametric form. 1 mark

e. Find the shortest distance from the origin to the plane ψ. 2 marks

f. Find the coordinates of point D. 2 marks

DO NOT WRITE IN THIS AREA

Question 6 (9 marks)

A forest ranger wishes to investigate the mass of adult male koalas in a Victorian forest. A random sample of 20 such koalas has a sample mean of 11.39 kg.

It is known that the mass of adult male koalas in the forest is normally distributed with a standard deviation of 1 kg.

a. Find a 95% confidence interval for the population mean (the mean mass of all adult male koalas in the forest). Give your values correct to two decimal places. 1 mark

b. Sixty such random samples are taken and their confidence intervals are calculated.

In how many of these confidence intervals would the actual mean mass of all adult male koalas in the forest be expected to lie? 1 mark

The ranger wants to decrease the width of the 95% confidence interval by 60% to get a better estimate of the population mean.

c. How many adult male koalas should be sampled to achieve this? 1 mark

It is thought that the mean mass of adult male koalas in the forest is 12 kg. The ranger thinks that the true mean mass is less than this and decides to apply a one-tailed statistical test. A random sample of 40 adult male koalas is taken and the sample mean is found to be 11.6 kg.

d. Write down the null hypothesis, H_0, and the alternative hypothesis, H_1, for the test. 1 mark

The ranger decides to apply the one-tailed test at the 1% level of significance and assumes the mass of adult male koalas in the forest is normally distributed with a mean of 12 kg and a standard deviation of 1 kg.

e. **i.** Find the p value for the test correct to four decimal places. 1 mark

ii. Draw a conclusion about the null hypothesis in **part d.** from the p value found above, giving a reason for your conclusion. 1 mark

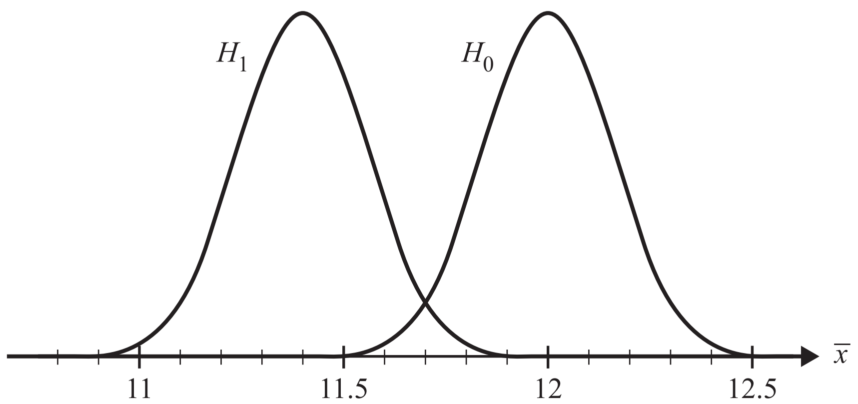
f. What is the critical sample mean (the smallest sample mean for H_0 not to be rejected) in this test? Give your answer in kilograms correct to three decimal places. 1 mark

Suppose that the true mean mass of adult male koalas in the forest is 11.4 kg, and the standard deviation is 1 kg. The level of significance of the test is still 1%.

g. What is the probability, correct to three decimal places, of the ranger making a type II error in the statistical test? 1 mark

h. The frequency curves for the sampling distributions associated with H_0 and H_1 are shown below.

Label the critical sample mean on the diagram and shade the region that represents the type II error. 1 mark



END OF QUESTION AND ANSWER BOOK

VICTORIAN CURRICULUM
AND ASSESSMENT AUTHORITY

Victorian Certificate of Education
2023

SPECIALIST MATHEMATICS

Written examination 2

FORMULA SHEET

Instructions

This formula sheet is provided for your reference.
A question and answer book is provided with this formula sheet.

Students are NOT permitted to bring mobile phones and/or any other unauthorised electronic devices into the examination room.

Mensuration

area of a circle segment	$\frac{r^2}{2}(\theta - \sin(\theta))$	volume of a sphere	$\frac{4}{3}\pi r^3$
volume of a cylinder	$\pi r^2 h$	area of a triangle	$\frac{1}{2}bc\sin(A)$
volume of a cone	$\frac{1}{3}\pi r^2 h$	sine rule	$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$
volume of a pyramid	$\frac{1}{3}Ah$	cosine rule	$c^2 = a^2 + b^2 - 2ab\cos(C)$

Algebra, number and structure (complex numbers)

$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = r\text{cis}(\theta)$	$\lvert z\rvert = \sqrt{x^2 + y^2} = r$	
$-\pi < \text{Arg}(z) \le \pi$	$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2)$	
$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$	de Moivre's theorem	$z^n = r^n \text{cis}(n\theta)$

Data analysis, probability and statistics

for independent random variables $X_1, X_2 \ldots X_n$	$\text{E}(aX_1 + b) = a\,\text{E}(X_1) + b$ $\text{E}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1\text{E}(X_1) + a_2\text{E}(X_2) + \ldots + a_n\text{E}(X_n)$	
	$\text{Var}(aX_1 + b) = a^2\text{Var}(X_1)$ $\text{Var}(a_1X_1 + a_2X_2 + \ldots + a_nX_n)$ $= a_1^2\text{Var}(X_1) + a_2^2\text{Var}(X_2) + \ldots + a_n^2\text{Var}(X_n)$	
for independent identically distributed variables $X_1, X_2 \ldots X_n$	$\text{E}(X_1 + X_2 + \ldots + X_n) = n\mu$	
	$\text{Var}(X_1 + X_2 + \ldots + X_n) = n\sigma^2$	
approximate confidence interval for μ	$\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$	
distribution of sample mean $\bar{X}$	mean	$\text{E}(\bar{X}) = \mu$
	variance	$\text{Var}(\bar{X}) = \frac{\sigma^2}{n}$

Calculus

$\frac{d}{dx}(x^n) = nx^{n-1}$
$\frac{d}{dx}(e^{ax}) = ae^{ax}$
$\frac{d}{dx}(\log_e(x)) = \frac{1}{x}$
$\frac{d}{dx}(\sin(ax)) = a\cos(ax)$
$\frac{d}{dx}(\cos(ax)) = -a\sin(ax)$
$\frac{d}{dx}(\tan(ax)) = a\sec^2(ax)$
$\frac{d}{dx}(\cot(ax)) = -a\,\text{cosec}^2(ax)$
$\frac{d}{dx}(\sec(ax)) = a\sec(ax)\tan(ax)$
$\frac{d}{dx}(\text{cosec}(ax)) = -a\,\text{cosec}(ax)\cot(ax)$
$\frac{d}{dx}(\sin^{-1}(ax)) = \frac{a}{\sqrt{1-(ax)^2}}$
$\frac{d}{dx}(\cos^{-1}(ax)) = \frac{-a}{\sqrt{1-(ax)^2}}$
$\frac{d}{dx}(\tan^{-1}(ax)) = \frac{a}{1+(ax)^2}$

$\int x^n dx = \frac{1}{n+1}x^{n+1} + c, \quad n \neq -1$
$\int e^{ax} dx = \frac{1}{a}e^{ax} + c$
$\int \frac{1}{x} dx = \log_e \lvert x \rvert + c$
$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax) + c$
$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$
$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$
$\int \operatorname{cosec}^2(ax)dx = -\frac{1}{a}\cot(ax) + c$
$\int \sec(ax)\tan(ax)\,dx = \frac{1}{a}\sec(ax) + c$
$\int \operatorname{cosec}(ax)\cot(ax)\,dx = -\frac{1}{a}\operatorname{cosec}(ax) + c$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + c,\ a > 0$
$\int \frac{a}{a^2 + x^2} dx = \tan^{-1}\left(\frac{x}{a}\right) + c$
$\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c, \quad n \neq -1$
$\int \frac{1}{ax+b} dx = \frac{1}{a}\log_e \lvert ax+b \rvert + c$

Calculus – continued

product rule	$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$
quotient rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
chain rule	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
integration by parts	$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
Euler's method	If $\frac{dy}{dx} = f(x, y)$, $x_0 = a$ and $y_0 = b$, then $x_{n+1} = x_n + h$ and $y_{n+1} = y_n + h \times f(x_n, y_n)$.
arc length parametric	$\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area Cartesian about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$
surface area Cartesian about y-axis	$\int_{y_1}^{y_2} 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$
surface area parametric about x-axis	$\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$
surface area parametric about y-axis	$\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

Kinematics

acceleration	$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$	
constant acceleration formulas	$v = u + at$	$s = ut + \frac{1}{2}at^2$
	$v^2 = u^2 + 2as$	$s = \frac{1}{2}(u+v)t$

Vectors in two and three dimensions

$\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$	$\lvert\underset{\sim}{r}(t)\rvert = \sqrt{x(t)^2 + y(t)^2 + z(t)^2}$
	$\dot{\underset{\sim}{r}}(t) = \frac{d\underset{\sim}{r}}{dt} = \frac{dx}{dt}\underset{\sim}{i} + \frac{dy}{dt}\underset{\sim}{j} + \frac{dz}{dt}\underset{\sim}{k}$
for $\underset{\sim}{r}_1 = x_1\underset{\sim}{i} + y_1\underset{\sim}{j} + z_1\underset{\sim}{k}$ and $\underset{\sim}{r}_2 = x_2\underset{\sim}{i} + y_2\underset{\sim}{j} + z_2\underset{\sim}{k}$	vector scalar product $\underset{\sim}{r}_1 \cdot \underset{\sim}{r}_2 = \lvert\underset{\sim}{r}_1\rvert\lvert\underset{\sim}{r}_2\rvert\cos(\theta) = x_1x_2 + y_1y_2 + z_1z_2$
	vector cross product $\underset{\sim}{r}_1 \times \underset{\sim}{r}_2 = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1)\underset{\sim}{i} + (x_2z_1 - x_1z_2)\underset{\sim}{j} + (x_1y_2 - x_2y_1)\underset{\sim}{k}$
vector equation of a line	$\underset{\sim}{r}(t) = \underset{\sim}{r}_1 + t\underset{\sim}{r}_2 = (x_1 + x_2t)\underset{\sim}{i} + (y_1 + y_2t)\underset{\sim}{j} + (z_1 + z_2t)\underset{\sim}{k}$
parametric equation of a line	$x(t) = x_1 + x_2t \quad y(t) = y_1 + y_2t \quad z(t) = z_1 + z_2t$
vector equation of a plane	$\underset{\sim}{r}(s, t) = \underset{\sim}{r}_0 + s\underset{\sim}{r}_1 + t\underset{\sim}{r}_2$ $= (x_0 + x_1s + x_2t)\underset{\sim}{i} + (y_0 + y_1s + y_2t)\underset{\sim}{j} + (z_0 + z_1s + z_2t)\underset{\sim}{k}$
parametric equation of a plane	$x(s, t) = x_0 + x_1s + x_2t,\ y(s, t) = y_0 + y_1s + y_2t,\ z(s, t) = z_0 + z_1s + z_2t$
Cartesian equation of a plane	$ax + by + cz = d$

Circular functions

$\cos^2(x) + \sin^2(x) = 1$	
$1 + \tan^2(x) = \sec^2(x)$	$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$
$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$	$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$	$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$	$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$
$\sin(2x) = 2\sin(x)\cos(x)$	
$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$	$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$
$\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$	$\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$

END OF FORMULA SHEET