VCE Specialist Mathematics

Written examination 2 – End of year

Sample questions

These sample questions are intended to demonstrate how new aspects of Units 3 and 4 of VCE Specialist Mathematics may be examined in written examination 2. They do **not** constitute a full examination paper.

SECTION A – Multiple-choice questions

Question 1

Consider the following statement.

‘For all integers n, if n^2 is even, then n is even.’

Which one of the following is the contrapositive of this statement?

A. For all integers n, if n^2 is odd, then n is odd.

B. There exists an integer n such that n^2 is even and n is odd.

C. There exists an integer n such that n is even and n^2 is odd.

D. For all integers n, if n is odd, then n^2 is odd.

E. For all integers n, if n is even, then n^2 is even.

Question 2

The procedure below has been written in pseudocode.

```
declare integer n
declare integer f
declare integer t1
declare integer t2
set f to 0
set t1 to 2
set t2 to 3
set n to 3
repeat n times
       f = t1 + 2 × t2
       t2 = f
       print f
end loop
```

The output of the pseudocode is a list of numbers.

The final number in the list is

A. 3
B. 18
C. 38
D. 72
E. 78

Question 3

A vector perpendicular to both of the lines represented by $\underset{\sim}{r}_1 = 2\underset{\sim}{i} + 3\underset{\sim}{j} + t\left(\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$ and $\underset{\sim}{r}_2 = 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} + t\left(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}\right)$ is given by

A. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & 0 \\ 3 & 1 & -2 \end{vmatrix}$

B. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 2 & 3 & 0 \\ 2 & 1 & -1 \end{vmatrix}$

C. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 1 & -2 \\ 2 & 3 & 0 \end{vmatrix}$

D. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 3 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}$

E. $\begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ 1 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$

Question 4

Consider two points with coordinates (5, −6, 4) and (−3, −1, −10).

Which one of the following is the equation of the straight line that passes through these two points?

A. $\underset{\sim}{r}(t) = -3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k} + t\left(8\underset{\sim}{i} - 5\underset{\sim}{j} + 14\underset{\sim}{k}\right)$

B. $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k} + t\left(3\underset{\sim}{i} + \underset{\sim}{j} + 10\underset{\sim}{k}\right)$

C. $\underset{\sim}{r}(t) = -3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k} + t\left(5\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k}\right)$

D. $\underset{\sim}{r}(t) = 5\underset{\sim}{i} - 6\underset{\sim}{j} + 4\underset{\sim}{k} + t\left(-3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k}\right)$

E. $\underset{\sim}{r}(t) = 8\underset{\sim}{i} - 5\underset{\sim}{j} + 14\underset{\sim}{k} + t\left(-3\underset{\sim}{i} - \underset{\sim}{j} - 10\underset{\sim}{k}\right)$

Question 5

A plane is perpendicular to the vector $\underset{\sim}{n} = \underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$ and passes through the point (3, 2, −4).

The Cartesian equation of this plane is

A. $3x + 2y - 4z = -11$

B. $-x + y - 3z = 11$

C. $-3x - 2y + 4z = -11$

D. $x - y + 3z = 11$

E. $x - y + 3z = 3$

Question 6

The shortest distance between the planes given by $5x - 4y - 12z = 10$ and $-15x + 12y + 36z = 20$ is

A. 0

B. $\dfrac{10}{3\sqrt{185}}$

C. $\dfrac{10}{\sqrt{185}}$

D. $\dfrac{50}{3\sqrt{185}}$

E. $\dfrac{50}{\sqrt{185}}$

Question 7

The time taken by a machine to make electronic components varies normally with a mean of 20 seconds and a standard deviation of 2 seconds. After the machine is serviced, it is believed that the mean time taken has been reduced to 18.5 seconds with the standard deviation remaining the same.

A statistical test is proposed to check whether there is any evidence of a 1.5 second reduction in the mean time taken to make components. The test statistic will be the mean time taken to make a random sample of 16 such components. The type I error for the test will be $\alpha = 5\%$ with a critical sample mean of 19.2 seconds.

The probability of a type II error (β) for the test is closest to

A. 8%

B. 34%

C. 36%

D. 46%

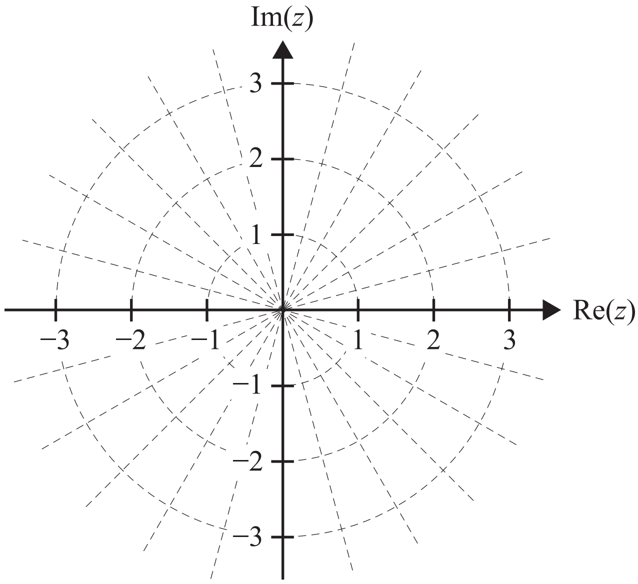
E. 95%

SECTION B

Question 1 (10 marks)

a. Express $\left\{z : |z| = \left|z - 2\operatorname{cis}\left(\frac{\pi}{4}\right)\right|,\ z \in C\right\}$ in the form $y = ax + b$, where $a, b \in R$. 2 marks

b. On the Argand diagram below, sketch and label $A = \{z : z\bar{z} = 4,\ z \in C\}$ and sketch and label $B = \left\{z : |z| = \left|z - 2\operatorname{cis}\left(\frac{\pi}{4}\right)\right|,\ z \in C\right\}$. Label the axis intercepts of the graph of B. 3 marks



c. On the Argand diagram in **part b.**, shade the region defined by
$\{z : z\bar{z} \le 4,\ z \in C\} \cap \{z : \operatorname{Re}(z) + \operatorname{Im}(z) \ge \sqrt{2},\ z \in C\}$. 1 mark

d. Find the area of the shaded region in **part c.** 2 marks

e. The elements of $\{z : z\bar{z} \le 4,\ z \in C\} \cap \left\{z : |z| = \left|z - 2\operatorname{cis}\left(\frac{\pi}{4}\right)\right|,\ z \in C\right\}$ provide two of the cube roots of w, where $w \in C$.

Write down all three cube roots of w in the form $r\operatorname{cis}(\theta)$ and find w in the form $a + ib$, where $a, b \in R$. 2 marks

Question 2 (10 marks)

In a certain region, 500 rare butterflies are released to maintain the species.

It is believed that the region can support a maximum of 30 000 such butterflies.

The butterfly population, P, t years after release can be modelled by the logistic differential equation $\frac{dP}{dt} = rP\left(1 - \frac{P}{30000}\right)$, where r is the growth rate of the population.

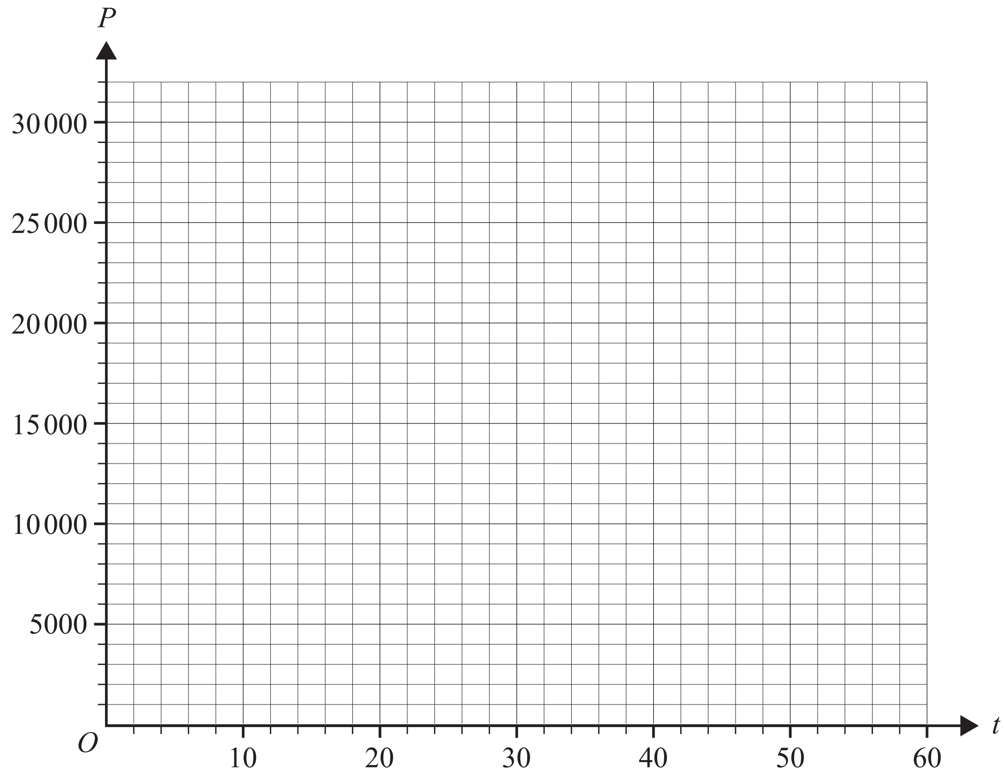
a. Use an integration technique and partial fractions to solve the differential equation above to find P in terms of r and t. 3 marks

b. Given that after 10 years there are 1930 butterflies in the population, find the value of r correct to two decimal places. 2 marks

c. What is the initial rate of increase of the population, correct to one decimal place? 1 mark

d. After how many years will the population reach 10 000 butterflies? Give your answer correct to one decimal place. 1 mark

e. Sketch the graph of P versus t on the axes below, showing the value of the vertical intercept. Label the point of fastest population growth as a coordinate pair (t, P), with t labelled correct to two decimal places, and label the asymptote with its equation. 3 marks



Question 3 (10 marks)

A plane, Π_1, is described by the parametric equations

$$x = 1 + 2s + 3t$$
$$y = -2 - s - 2t$$
$$z = 2 - s + t$$

A second plane, Π_2, contains the point $P(1, 0, 3)$ and is parallel to the plane Π_1.

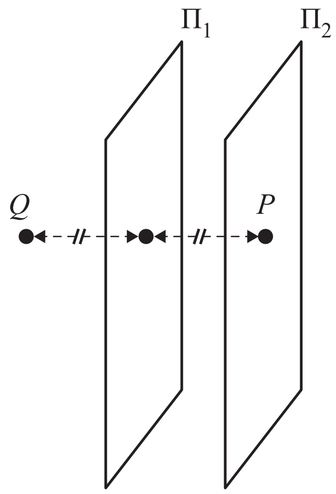
a. Find a vector equation of the plane Π_1 in the form $\underset{\sim}{r} = \underset{\sim}{a} + s\underset{\sim}{b} + t\underset{\sim}{c}$. 2 marks

b. Hence, find a Cartesian equation of the plane Π_1. 2 marks

c. Find a Cartesian equation of the plane Π_2. 1 mark

d. **i.** Find the shortest distance between the planes Π_1 and Π_2. 2 marks

ii.



Hence, find the coordinates of point Q, which is the reflection of point P in the plane Π_1, as shown in the diagram above. 3 marks

Question 4 (10 marks)

a. Find the shortest distance between the two parallel lines given by $\underset{\sim}{r}(t) = 4\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + t\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$, where $t \in R$, and $\underset{\sim}{r}(s) = 5\underset{\sim}{i} + 4\underset{\sim}{j} - 2\underset{\sim}{k} + s\left(-\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}\right)$, where $s \in R$. 3 marks

b. Given that the lines with equations $\underset{\sim}{r}(t) = \underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k} + t\left(3\underset{\sim}{i} + 5\underset{\sim}{j} - a\underset{\sim}{k}\right)$, where $t \in R$, and $\underset{\sim}{r}(s) = -6\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} + s\left(4\underset{\sim}{i} - 10\underset{\sim}{j} + 6\underset{\sim}{k}\right)$, where $s \in R$, intersect, find the value of a and the point of intersection. 4 marks

c. The line with equation $\underset{\sim}{r}(t) = \underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k} + t\left(4\underset{\sim}{i} + b\underset{\sim}{j} + 2\underset{\sim}{k}\right)$, where $t, b \in R$, is parallel to the plane with equation $2x - 3y - z = 2$.

Find the value of b and the shortest distance of the line from the plane. 3 marks

Question 5 (10 marks)

a. Given the points $A(1, 0, 2)$, $B(2, 3, 0)$ and $C(1, 2, 1)$

i. find the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ 1 mark

ii. show that the Cartesian equation of the plane Π_1, containing the points A, B and C, is $x + y + 2z = 5$. 1 mark

b. A second plane, Π_2, has the Cartesian equation $x - y - z = 0$.
L is the line of intersection of the planes Π_1 and Π_2.

i. Find the coordinates of the point P, where L crosses the y-z plane. 1 mark

ii. Hence, find the vector equation of the line L. 2 marks

iii. Find the distance from the point A to the plane Π_2. 2 marks

iv. Find the distance from the point A to the line L. 3 marks

Question 6 (11 marks)

The position vector $\underset{\sim}{r}_S(t)$, from an origin O, of a sparrow t seconds after being sighted is modelled by $\underset{\sim}{r}_S(t) = 23t\,\underset{\sim}{i} + 5t\,\underset{\sim}{j} + \left(4\sqrt{2}\sin\left(\frac{\pi t}{2}\right) + 4\sqrt{2}\right)\underset{\sim}{k}$, $t \geq 0$, where $\underset{\sim}{i}$ is a unit vector in the forward direction, $\underset{\sim}{j}$ is a unit vector to the left and $\underset{\sim}{k}$ is a unit vector vertically up. Displacement components are measured in centimetres.

a. Find the value of t when the sparrow first lands on the ground. 2 marks

b. Find the distance of the sparrow from O when it first lands. Give your answer correct to one decimal place. 2 marks

c. Find the maximum flight speed, in centimetres per second, of the sparrow. Give your answer correct to one decimal place. 2 marks

A second bird, a miner, flies such that its velocity vector $\underset{\sim}{v}_M(t)$, relative to the same origin O, is modelled by $\underset{\sim}{v}_M(t) = 6\underset{\sim}{i} + \underset{\sim}{j} + \left(\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)\right)\underset{\sim}{k}$, $t \geq 0$, where velocity components are measured in centimetres per second.

d. Given that the miner has an initial position vector of $10\underset{\sim}{i} + 4\underset{\sim}{j} + 4\sqrt{2}\underset{\sim}{k}$, show that its position vector at time t seconds is given by $\underset{\sim}{r}_M(t) = (6t + 10)\underset{\sim}{i} + (t+4)\underset{\sim}{j} + \left(\sin\left(\frac{\pi t}{6}\right) + 4\sqrt{2}\right)\underset{\sim}{k}$. 2 marks

e. The sparrow and the miner are at the same position at different times.

Find the coordinates of this position and the times at which each bird is at this position. 3 marks

Answers to multiple-choice questions

Question	Answer
1	D
2	C
3	E
4	A
5	B
6	D
7	A